Chaotic Hydromagnetic Convection

Wiesław M. Macek[1,2]
[1] Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University, Wóycickiego 1/3, 01-938 Warsaw, Poland;
[2] Space Research Centre, Polish Academy of Sciences, Bartycka 18 A, 00-716 Warsaw, Poland
e-mail: macek@cbk.waw.pl, http://www.cbk.waw.pl/∼macek

PRE, Aug. 2010

Abstract

We consider convection in a horizontally magnetized viscous fluid layer in the gravitational field heated from below with a vertical temperature gradient. Following Rayleigh-Bénard scenario and using a general magnetohydrodynamic approach, we obtain a simple set of four ordinary differential equations. In addition to the usual three-dimensional Lorenz model a new variable describes the profile of the induced magnetic field. We show that nonperiodic oscillations are influenced by anisotropic magnetic forces resulting not only in an additional viscosity but also substantially modifying nonlinear forcing of the system. On the other hand, this can stabilize convective motion of the flow. However, for certain values of the model parameters we have identified a deterministic intermittent behavior of the system resulting from bifurcation. In this way, we have identified here a basic mechanism of intermittent release of energy bursts, which is frequently observed in space and laboratory plasmas. Hence, we propose this model as a useful tool for the analysis of intermittent behavior of various environments, including convection in planets and stars. Therefore, we hope that our simple but still a more general nonlinear model could shed light on the nature of hydromagnetic convection.

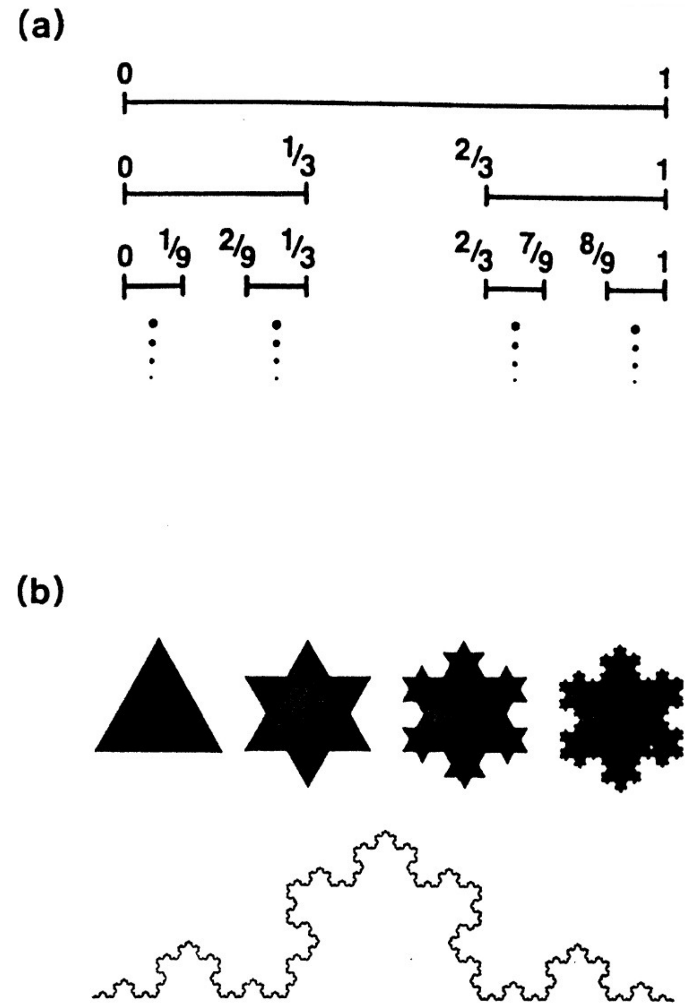
Plan of Presentation

1. Introduction
   - Fractals
   - Attracting and Stable Points
   - Strange Attractors and Deterministic Chaos
   - Intermittency
   - Rayleigh-Bénard Convection and Lorenz Model
2. Derivation of the Model of Hydromagnetic Convection
   - Basic Equations
   - Approximations
   - Generalized Lorenz Model for a Magnetized Fluid
3. Analysis of the Model
   - Dynamical Behavior in Control Parameters Space
   - New Strange Attractors
   - Intermittency
4. Possible Applications
5. Conclusions

Fractals

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole.

Fractals are generally *self-similar* and independent of scale (fractal dimension).

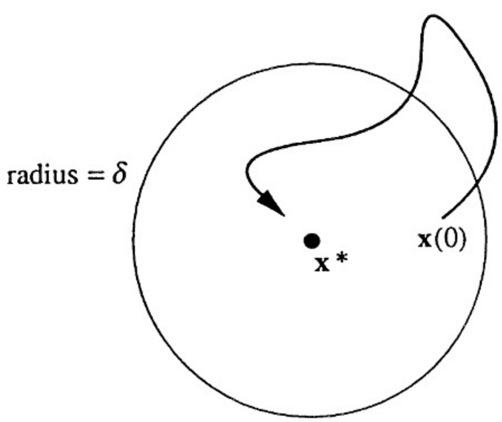


Attracting and Stable Fixed Points

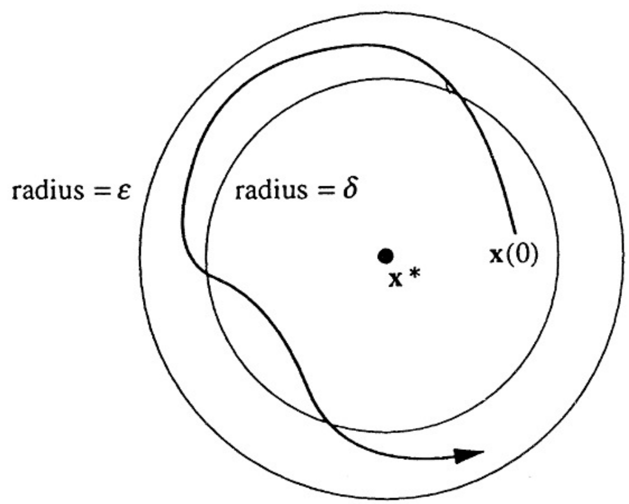
We consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, where $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$.

We say that $\mathbf{x}^*$ is *attracting* if there is a $\delta > 0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$ whenever $|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$: any trajectory that starts within a distance δ of $\mathbf{x}^*$ is guaranteed to converge to $\mathbf{x}^*$.

A fixed point $\mathbf{x}^*$ is *Lyapunov stable* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$ whenever $t \geq 0$ and $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$: all trajectories that start within δ of $\mathbf{x}^*$ remain within ε of $\mathbf{x}^*$ for all positive time.



Attracting



Liapunov stable

Attractors

An **ATTRACTOR** is a *closed* set A with the properties:

1. A is an INVARIANT SET:
   any trajectory $\mathbf{x}(t)$ that start in A stays in A for ALL time t.
2. A ATTRACTS AN OPEN SET OF INITIAL CONDITIONS:
   there is an open set U containing A ($\subset U$) such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$.
3. A is MINIMAL:
   there is NO proper subset of A that satisfies conditions 1 and 2.

STRANGE ATTRACTOR is an attracting set that is a fractal: has zero measure in the embedding phase space and has FRACTAL dimension. Trajectories within a strange attractor appear to skip around randomly.

Dynamics on **CHAOTIC ATTRACTOR** exhibits sensitive (exponential) dependence on initial conditions (the 'butterfly' effect).

Deterministic Chaos

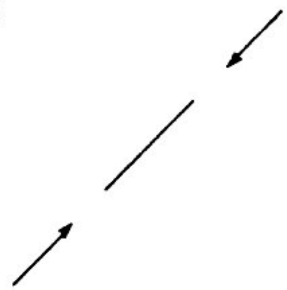
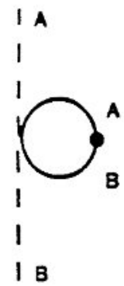
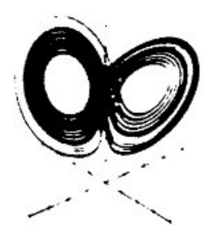
CHAOS ($\chi\alpha o\varsigma$) is

- NON-PERIODIC long-term behavior
- in a DETERMINISTIC system
- that exhibits SENSITIVITY TO INITIAL CONDITIONS.

We say that a bounded solution $\mathbf{x}(t)$ of a given dynamical system is SENSITIVE TO INITIAL CONDITIONS if there is a finite fixed distance $r > 0$ such that for any neighborhood $\|\Delta\mathbf{x}(0)\| < \delta$, where $\delta > 0$, there exists (at least one) other solution $\mathbf{x}(t)$ + $\Delta\mathbf{x}(t)$ for which for some time $t \geq 0$ we have $\|\Delta\mathbf{x}(t)\| \geq r$.

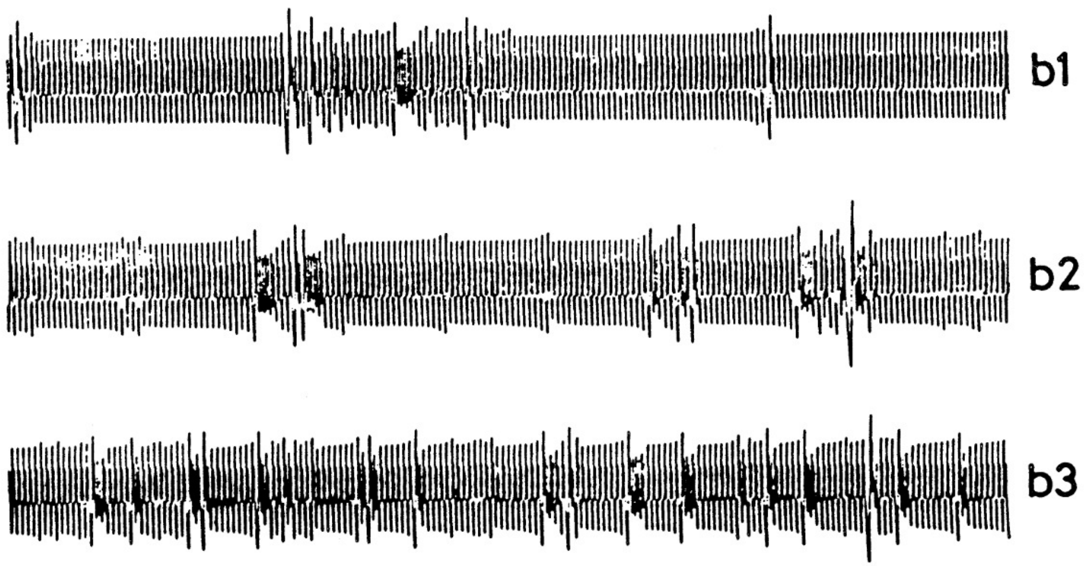
There is a fixed distance r such that no matter how precisely one specify an initial state there is a nearby state (at least one) that gets a distance r away.

Given $\mathbf{x}(\mathbf{t}) = \{x_1(t), \ldots, x_n(t)\}$ any positive finite value of Lyapunov exponents $\lambda_k = \lim_{t\to\infty} \frac{1}{t} \ln \left| \frac{\Delta x_k(t)}{\Delta x_k(0)} \right|$, where $k = 1, \ldots n$, implies chaos.

	ATTRACTOR	LYAPUNOV EXPONENT SPECTRUM
C 	LIMIT CYCLE 	(0,−,−)
D 	POINT 	(−,−,−)
E	STRANGE ATTRACTOR 	(+,0,−)

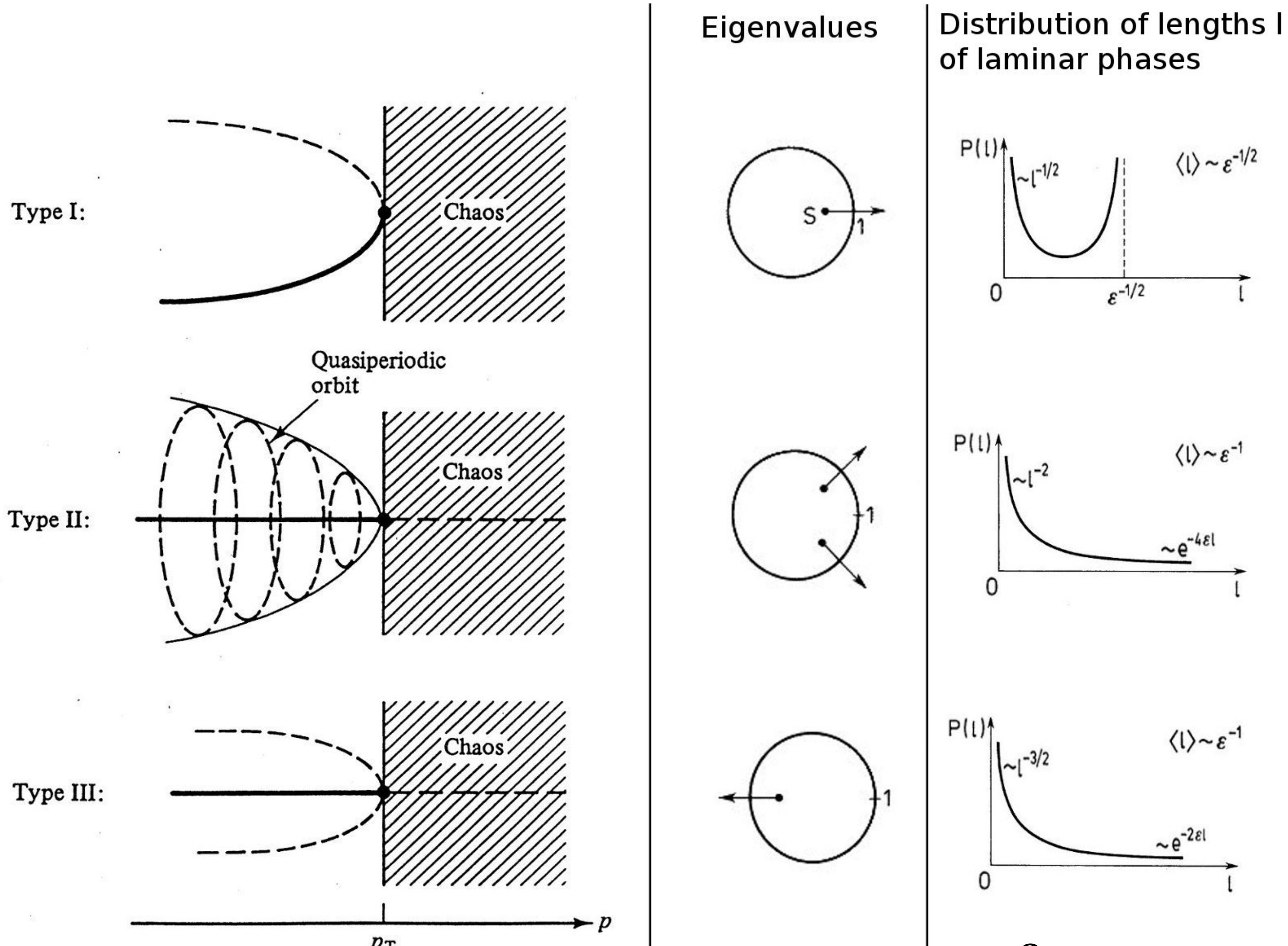
Intermittency

In dynamical systems theory: occurrence of a signal that alternates randomly between long periods of regular behavior and relatively short irregular bursts. In other words, motion in intermittent dynamical system is nearly periodic with occasional irregular bursts.



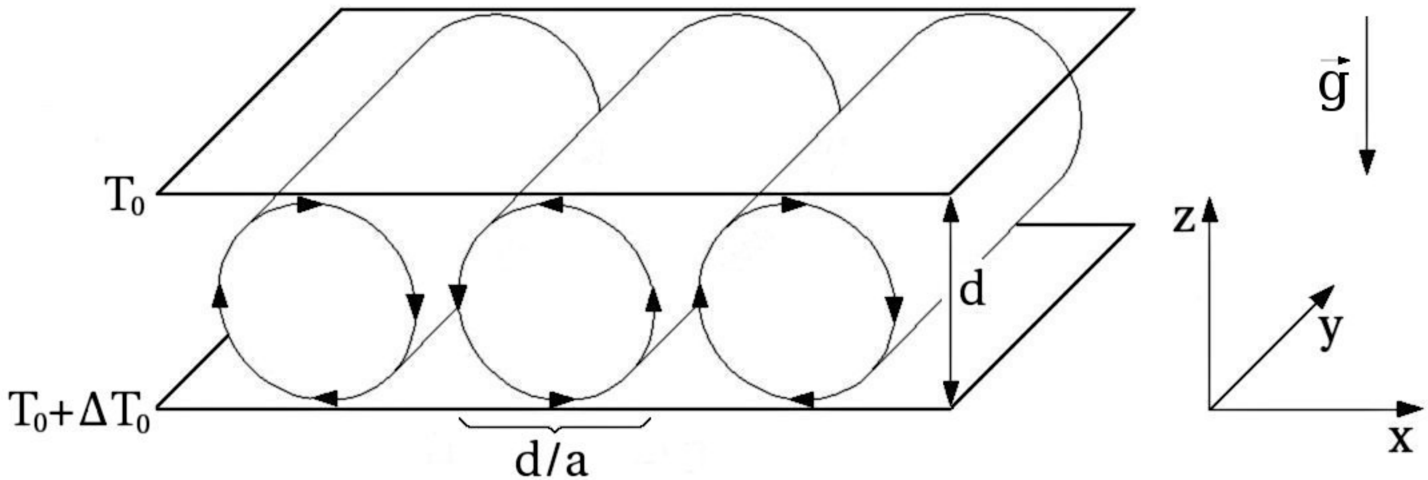
Pomeau & Manneville, 1980

Basic Types of Intermittency



$$\varepsilon = p - p_T$$

Rayleigh-Bénard Convection



Navier-Stokes equation

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \nu\,\Delta\mathbf{v} + \mathbf{g},$$

Heat conduction equation

$$\frac{dT}{dt} = \kappa\,\Delta T,$$

Continuity equation

$$\nabla\cdot\mathbf{v} = 0.$$

Rayleigh-Bénard Convection

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \nu\,\rho\Delta\mathbf{v} + \rho\mathbf{f}, \quad \frac{\mathrm{d}T}{\mathrm{d}t} = \kappa\,\Delta T, \quad \nabla\cdot\mathbf{v} = 0 \quad (\nabla\times\ldots)$$

time and space changes: $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$ for incompressible fluid

Oberbeck-Boussinesq approximation:

$\rho = \rho_o$ except for $\rho = \rho_o(1-\beta\delta T)$ in the buoyancy term $\rho\mathbf{g}$ ($\mathbf{f} = \mathbf{g}$, z-axis)

$$\mathbf{v} = \nabla\times\Psi \qquad \Psi = \{0, \psi(x,z,t), 0\}$$

$$T(x,z,t) = T(x,0,t) + \delta T_0 - \frac{\delta T_0}{d}z + \theta(x,z,t)$$

Bispectral representation:

$$\frac{a}{\kappa(1+a^2)}\psi(x,z,t) = X(t)\sqrt{2}\sin\left(\frac{\pi a}{d}x\right)\sin\left(\frac{\pi}{d}z\right)$$

$$\frac{\pi R_a}{R_c\delta T_0}\theta(x,z,t) = Y(t)\sqrt{2}\cos\left(\frac{\pi a}{d}x\right)\sin\left(\frac{\pi}{d}z\right) - Z(t)\sin\left(\frac{2\pi}{d}z\right)$$

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

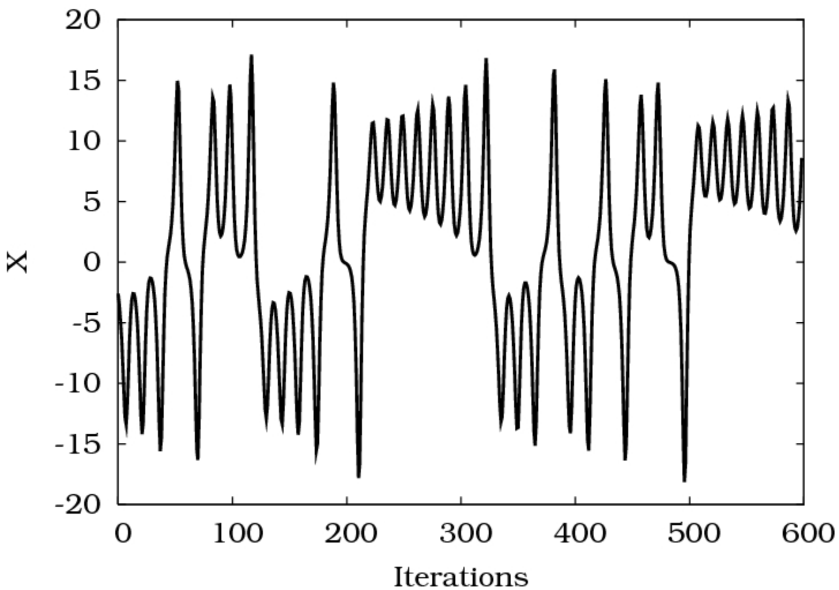
The feasibility of very-long-range weather prediction is examined in the light of these results.

Lorenz Model

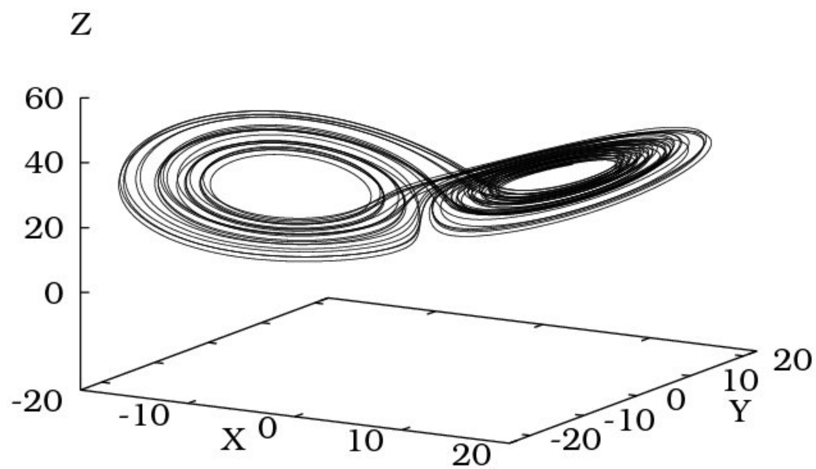
$$\dot{X} = \sigma(Y - X)$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ$$

Parameters:
$r = 28$, $\sigma = 10$, $b = 8/3$

Time series for X



Attractor



PHYSICAL REVIEW E **82**, 027301 (2010)

Model for hydromagnetic convection in a magnetized fluid

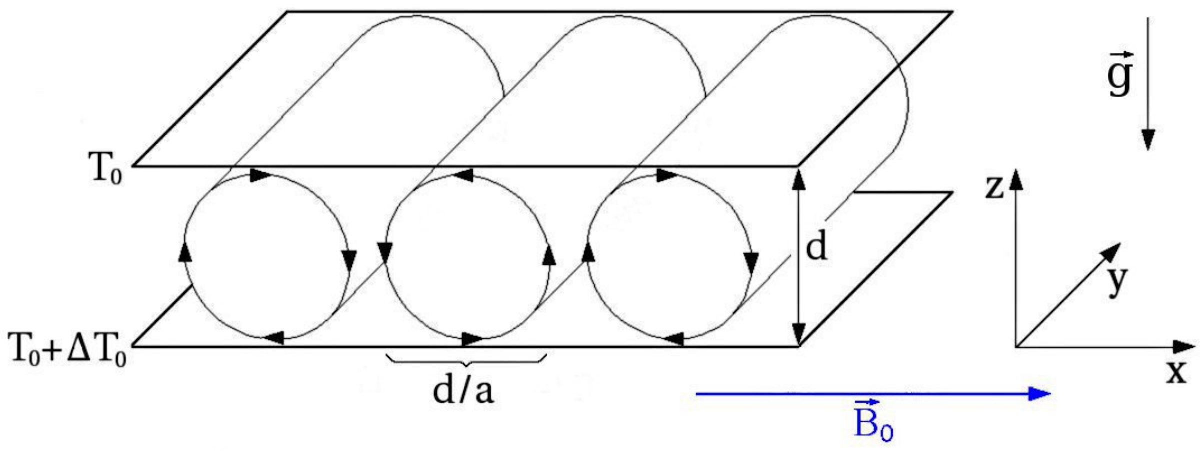
Wiesław M. Macek*
Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University, Warsaw, Poland

Marek Strumik†
Space Research Centre, Polish Academy of Sciences, Warsaw, Poland
(Received 23 April 2010; published 4 August 2010)

We consider convection in a horizontally magnetized viscous fluid layer in the gravitational field heated from below with a vertical temperature gradient. Following Rayleigh-Bénard scenario and using a general magnetohydrodynamic approach, we obtain a simple set of four ordinary differential equations. In addition to the usual three-dimensional Lorenz model a new variable describes the profile of the induced magnetic field. We show that nonperiodic oscillations are influenced by anisotropic magnetic forces resulting not only in an additional viscosity but also substantially modifying nonlinear forcing of the system. On the other hand, this can stabilize convective motion of the flow. However, for certain values of the model parameters we have identified a deterministic intermittent behavior of the system resulting from bifurcation. In this way, we have identified here a basic mechanism of intermittent release of energy bursts, which is frequently observed in space and laboratory plasmas. Hence, we propose this model as a useful tool for the analysis of intermittent behavior of various environments, including convection in planets and stars. Therefore, we hope that our simple but still a more general nonlinear model could shed light on the nature of hydromagnetic convection.

DOI: 10.1103/PhysRevE.82.027301 PACS number(s): 47.20.Ky, 47.65.−d, 82.40.Bj, 92.60.hk

Convection in a Magnetized Fluid



$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho}\nabla\left(p + \frac{\mathbf{B}^2}{2\mu_o}\right) + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu_o\rho} + \nu \triangle \mathbf{v} + \mathbf{f},$$

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B}\cdot\nabla)\mathbf{v} + \eta \triangle \mathbf{B},$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \kappa \triangle T,$$

Convection in a Magnetized Fluid

Additional conditions

$$\nabla \cdot \mathbf{v} = 0,$$
$$\nabla \cdot \mathbf{B} = 0$$

allow to define

$$\mathbf{v} = \nabla \times \Psi,$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

a stream (potential) function ψ for the flow $\mathbf{v}$:

$$\Psi = \{0, \psi(x, z, t), 0\}$$

and a (vector) potential $\mathbf{A}$ for the embedded magnetic field $\mathbf{B}$:

$$\mathbf{A}/(\mu_o \rho_o)^{1/2} \;=\; \{0, \alpha(x, z, t) - \upsilon_{\mathrm{A}o} z, 0\}$$

where $\upsilon_{\mathrm{A}o} = B_o/(\mu_o \rho_o)^{1/2}$ is the initial Alfvén speed.

Approximation used: $(\mathbf{B} \cdot \nabla)\mathbf{v} \approx (\mathbf{B_o} \cdot \nabla)\mathbf{v}$.

Rayleigh-Bénard Convection in a Magnetized Fluid

Double asymmetric Fourier representation:

$$\psi(x,z,t) = \sqrt{2}\,\frac{1+a^2}{a}\,\kappa\, X(t)\ \sin\left(\frac{\pi a}{h}x\right)\ \sin\left(\frac{\pi}{h}z\right),$$

$$\theta(x,z,t) = \frac{R_c\delta T_0}{\pi R_a}\left[\sqrt{2}\, Y(t)\cos\left(\frac{\pi a}{d}x\right)\sin\left(\frac{\pi}{d}z\right) - Z(t)\sin\left(\frac{2\pi}{d}z\right)\right],$$

$$\alpha(x,z,t) = \sqrt{2}\,\frac{1+a^2}{a}\,\kappa\, W(t)\ \cos\left(\frac{\pi a}{h}x\right)\ \sin\left(\frac{\pi}{h}z\right).$$

The velocity field is described by $\psi(x,z,t)$ [variable $X(t)$],
the temperature gradient by $\theta(x,z,t)$ [variables $Y(t), Z(t)$],
and the induced magnetic field by $\alpha(x,z,t)$ [variable $W(t)$].

Lorenz Model for a Magnetized Fluid

Using those approximations fluid dynamics can be described by a simple set of four ordinary differential equations

$$\dot{X} = -\sigma X + \sigma Y - \omega_o W,$$
$$\dot{Y} = -XZ + rX - Y,$$
$$\dot{Z} = XY - bZ,$$
$$\dot{W} = (4\pi/\mu_o)\ \omega_o X - \sigma_{\mathrm{m}} W,$$

with respect to normalized time $t' = (1 + a^2)\ \kappa(\pi/h)^2\ t$,
using a geometrical factor $b = 4/(1 + a^2)$.

Control parameter $r = R_{\mathrm{a}}/R_{\mathrm{c}}$;
Rayleigh number $R_{\mathrm{a}} = g\beta h^3 \delta T/(\nu\kappa)$, critical number $R_{\mathrm{c}} = (1 + a^2)^3(\pi^2/a)^2$.

Magnetic control parameter
$\omega_o = \upsilon_{\mathrm{A}o}/\upsilon_o$;
$\upsilon_{\mathrm{A}o} = B_o/(\mu_o\rho_o)^{1/2}$, $\upsilon_o = 16\pi^2\kappa/(abh\mu_o)$

Prandtl number $\sigma = \nu/\kappa$;
Magnetic Prandtl number $\sigma_{\mathrm{m}} = \eta/\kappa$.

Lorenz Model for a Magnetized Fluid

Combining the set of the generalized Lorenz system we can write

$$\ddot{X} + \sigma\dot{X} + (\sigma r - (4\pi/\mu_o)\ \omega_o^2)X = -\sigma(Y + XZ) + \sigma_{\mathrm{m}}\omega_o W,$$

$$\ddot{W} + \sigma_{\mathrm{m}}\dot{W} + (4\pi/\mu_o)\ \omega_o^2 W = (4\pi/\mu_o)\ \sigma\omega_o(Y - X).$$

Hence formally both variables X and W satisfy the equations of two familiar damped linear oscillators with nonlinear driving forces. Moreover, we can see that the coupling between X, W and Y, Z is enhanced owing to the magnetic field $\mathbf{B}$. Obviously, when $\omega_o = 0$ this coupling ceases and the variable W is damped by the magnetic viscosity.

Fixed Points (Equilibrium)

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{F}(\mathbf{x}^*) = \mathbf{0}, \quad \mathbf{x}^* = \{X^*, Y^*, Z^*, W^*\}$$

$C^0 = \{0, 0, 0, 0\}$,
$C^{\pm} = \{\pm d/\sqrt{1+e}, \pm\, d\sqrt{(1+e)}, r-(1+e), \pm(\sigma/\omega_o)de/\sqrt{1+e}\}$,
where $d = \sqrt{b((r-1)-e)}$, and $e = (4\pi/\mu_o)\ \omega_o^2/(\sigma\ \sigma_{\rm m})$.

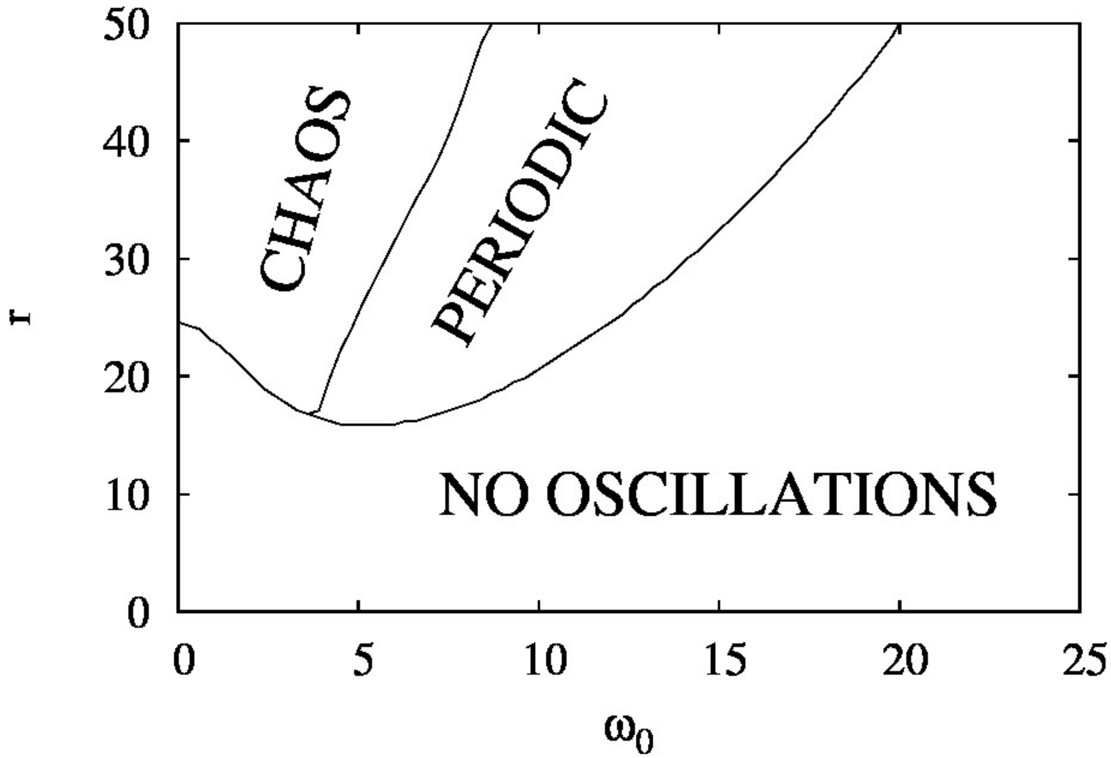
C^0 stable for $0 \le r < r_o$,
$C^{\pm}$ stable for $r_o \le r < r_H$,
$r_o = 1+e$ is a critical value for the onset of convection,
$r = r_H$ is a critical value, where a Hopf bifurcation takes place.

The critical number r_o for the onset of convection increases with the magnetic field, thus the magnetic field should stabilize the convection as regards to the appearance of convective rolls.

However, if we consider oscillations of the convection rolls, the influence of the magnetic field is more intricate.

Long-term Behavior Depending on Control Parameters

$$r = R_a/R_c, \quad R_a = g\beta h^3 \delta T/(\nu\kappa),\ R_c = (1+a^2)^3(\pi^2/a)^2$$
$$\omega_o = \upsilon_{Ao}/\upsilon_o, \quad \upsilon_{Ao} = B_o/(\mu_o\rho_o)^{1/2},\ \upsilon_o = 16\pi^2\kappa/(abh\mu_o)$$

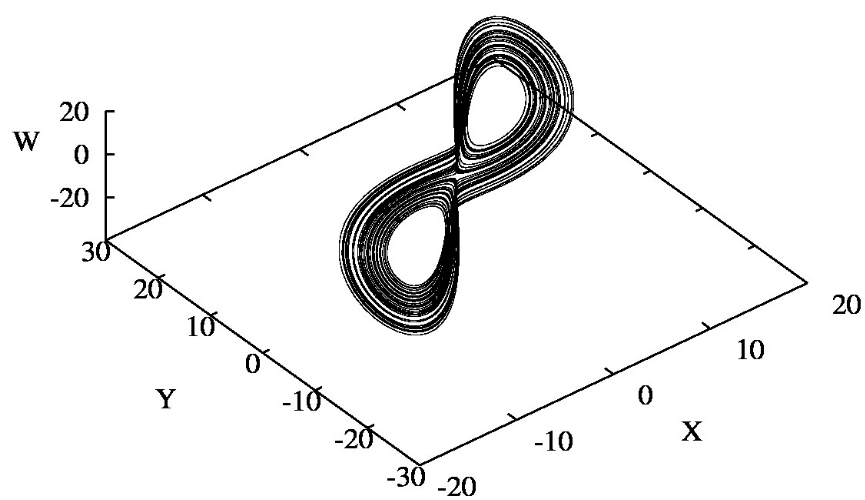


Strange Attractors for a Magnetized Fluid

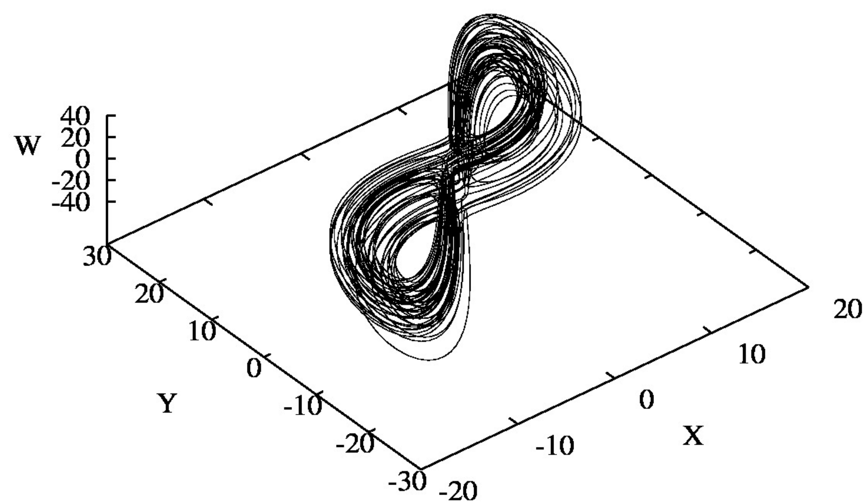
$$\dot{X} = -\sigma X + \sigma Y - \omega_o W,$$
$$\dot{Y} = -XZ + rX - Y,$$
$$\dot{Z} = XY - bZ,$$
$$\dot{W} = (4\pi/\mu_o)\ \omega_o X - \sigma_{\mathrm{m}} W,$$

Standard parameters: $r = 28$, $\sigma = 10$, and $b = 8/3$

$\omega_o = 1$, $\sigma_{\mathrm{m}} = 20$

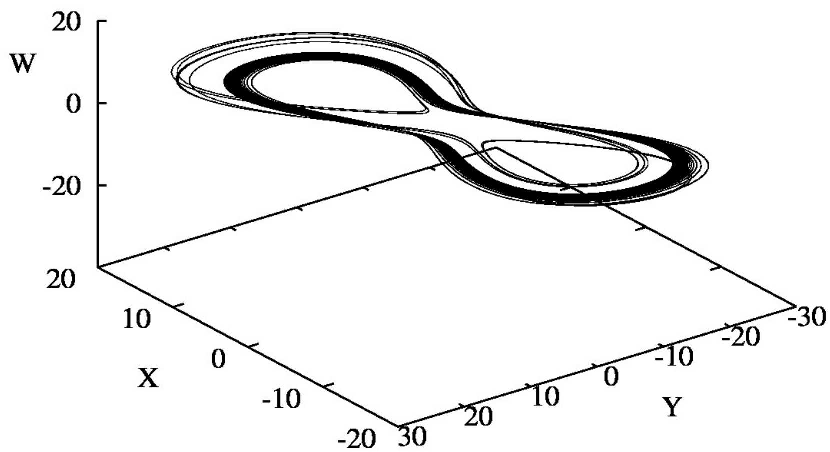


$\omega_o = 1$, $\sigma_{\mathrm{m}} \approx 0$

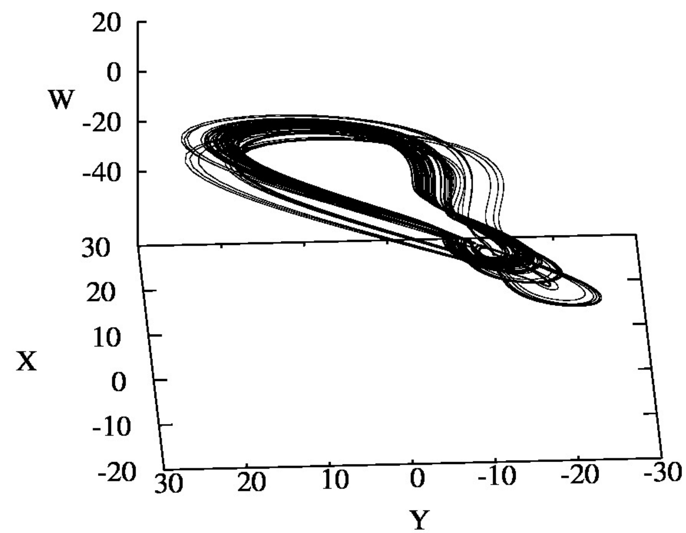


Strange Attractors for a Magnetized Fluid

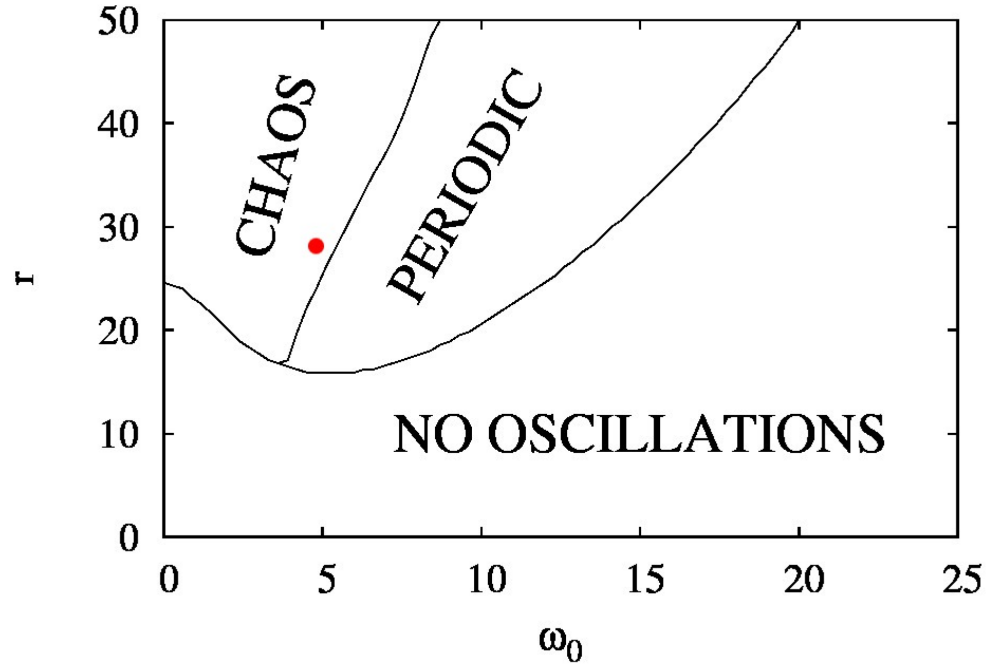
$\omega_o = 6,\ \sigma_m = 2$

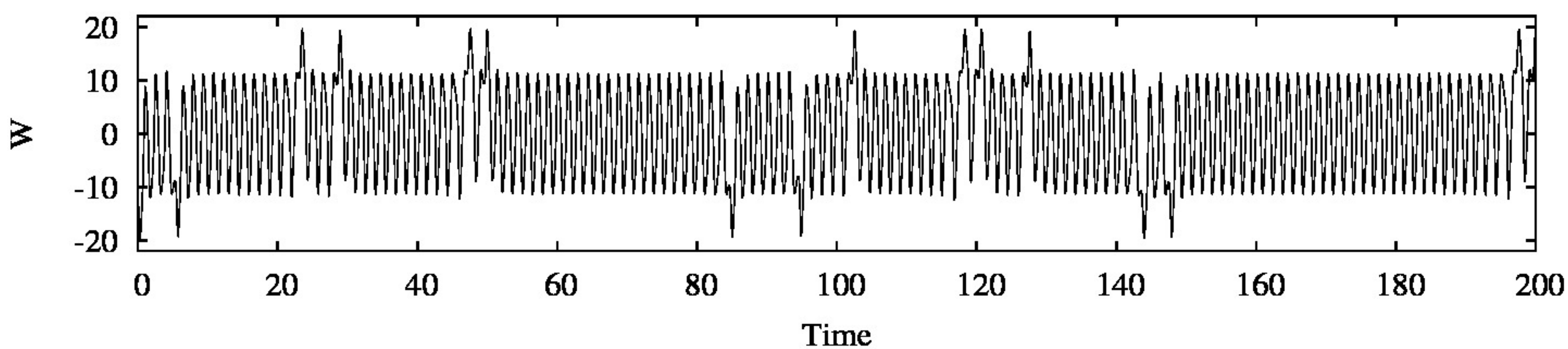


$\omega_o = 5,\ \sigma_m \approx 0$



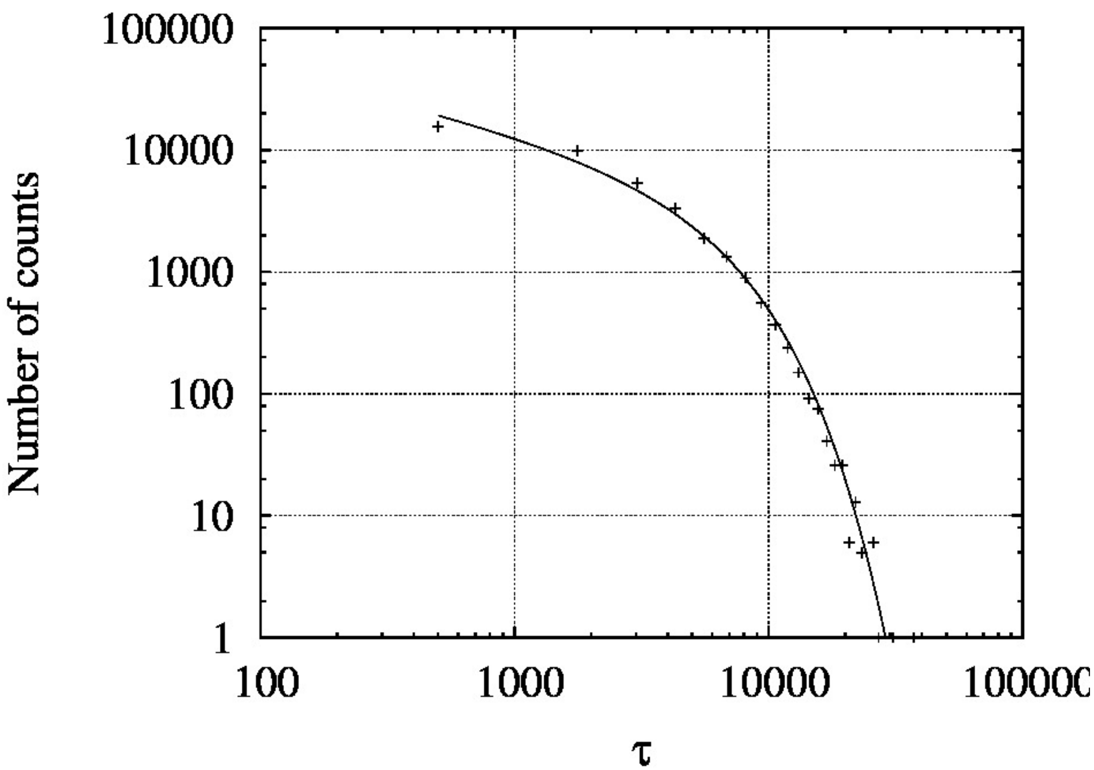
Intermittent Behavior





Intermittent Behavior

Type III intermittency (identified using Poincaré map based on Y values taken for X=0 plane crossings). Distribution of lengths τ of laminar phases for a deviation $\varepsilon = p - p_T > 0$ from a critical value for bifurcation p_T



$$P(\tau) \propto \frac{\varepsilon^{3/2} e^{4\varepsilon\tau}}{(e^{4\varepsilon\tau} - 1)^{3/2}}$$

$$\text{average} < \tau > \sim \varepsilon^{-1}$$

$$P(\tau) \sim \tau^{-3/2} \quad \text{for small} \quad \tau \to 0$$

(fully developed turbulence)

$$P(\tau) \sim e^{-2\varepsilon\tau} \quad \text{for large} \quad \tau \to \infty$$

(self-organized criticality)

Possible Applications of the Model

- liquid interiors of the Earth's core,
- interiors of the Sun and stars, including massive stars with heavy elements (*Brite* experiment),
- solar sunspots and coronal holes, granulation;
- the flow in the magnetosphere and heliosphere, and even in interstellar and intergalactic media;
- magneto-confined plasmas in tokamaks;
- nanodevices and microchannels in nanotechnolgy.

Conclusions

- We propose a new low-dimensional model describing self-consistently convective transport of magnetized fluids. In addition to the usual three-dimensional Lorenz model a new variable describes the profile of the induced magnetic field.

- It is clearly shown that the influence of the magnetic field is more intricate than purely stabilizing effect predicted by simple analytical models: increase of the magnetic field may both excite and damp oscillations. Also nonperiodic oscillations are influenced by anisotropic magnetic forces resulting not only in an additional viscosity but also substantially modifying nonlinear forcing of the system.

- At the boundary between chaotic and periodic region in control parameters space we have identified intermittent (type III) behavior. This feature is important from the experimental point of view.

One should note that, as is essential for intermittency, this transition from regular to irregular behavior results from the appearance and disappearance of fixed points or limit cycles and not from any stochastic forces.

- In this way, we have identified here a basic mechanism of intermittent release of energy bursts, $\nu|\mathbf{v}|^2 + \eta|\mathbf{B}|^2/(\mu_o\rho)$, which is frequently observed in space and laboratory plasmas.

- Hence we hope that our simple but still a more general nonlinear model could shed light on the nature of hydromagnetic turbulent convection, helping to identify chaotic and intermittent behavior in various environments.

- We propose this model as a useful tool for analysis of intermittent behavior of various environments, including convection in planets and stars.

References

[1] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[2] W. Macek and M. Strumik, Phys. Rev. E **82**, 027301 (2010).

[3] K. Rypdal and O. E. Garcia, Physics of Plasmas **14**, 022101 (2007).

[4] W. M. Macek, Physica D (Nonlinear Phenomena) **122**, 254 (1998).

[5] K. Rypdal and S. Ratynskaia, Phys. Rev. Lett. **94**, 225002 (2005).

[6] P. Bergé, Y. Pomeau, and C. Vidal, *Order within Chaos. Towards a deterministic approach to turbulence* (Viley, New York, 1984).

[7] L. D. Landau, E. M. Lifshits, and L. P. Pitaevskii, *Electrodynamics of continuous media*, vol. 8 (Pergamon Press, Oxford, 1984).

[8] Rayleigh, Lord, Phil. Mag. **32**, 529 (1916).

[9] H. G. Schuster, *Deterministic Chaos. An Introduction* (VCH Verlagsgesellschaft, Weinheim, 1988).

[10] A. Oberbeck, Annalen der Physik **243**, 271 (1879).

[11] J. Boussinesq, *Theorie Analystique de la Chaleur* (Gauthier-Villars, 1903).

[12] B. Saltzman, J. Atmos. Sci. **19**, 329 (1962).

[13] T. G. Cowling, *Magnetohydrodynamics* (Bristol: Adam Hilger, 1976).

[14] Y. Pomeau and P. Manneville, Commun. in Math. Phys. **74**, 189 (1980).