Multifractal Models for Solar Wind Turbulence

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Abstract

The question of multifractality is of great importance for the heliophysics because it allows us to look at intermittent turbulence in the solar wind. Starting from Richardson's scenario of turbulence, many authors try to recover the observed scaling exponents, using some simple and more advanced fractal and multifractal models of turbulence describing distribution of the energy flux between cascading eddies of various scales. In particular, the multifractal spectrum has been investigated using Voyager (magnetic field) data in the outer heliosphere and using Helios (plasma) data in the inner heliosphere. We have also analysed the spectrum for the solar wind attractor. The spectrum is found to be consistent with that for the multifractal measure of the self-similar weighted baker's map with two parameters describing uniform compression and natural invariant probability measure of the attractor of the system. In order to further quantify the multifractality, we also consider a generalized weighted Cantor set with two different scales describing nonuniform compression. We investigate the resulting multifractal spectrum depending on two scaling parameters and one probability measure parameter, especially for asymmetric scaling. We hope that this model will be a useful tool for analysis of intermittent turbulence in space plasmas.

Within the complex dynamics of the solar wind's fluctuating intermittent plasma parameters, there is a detectable, hidden order described by a generalized Cantor set which exhibits a multifractal structure.

Plan of Presentation

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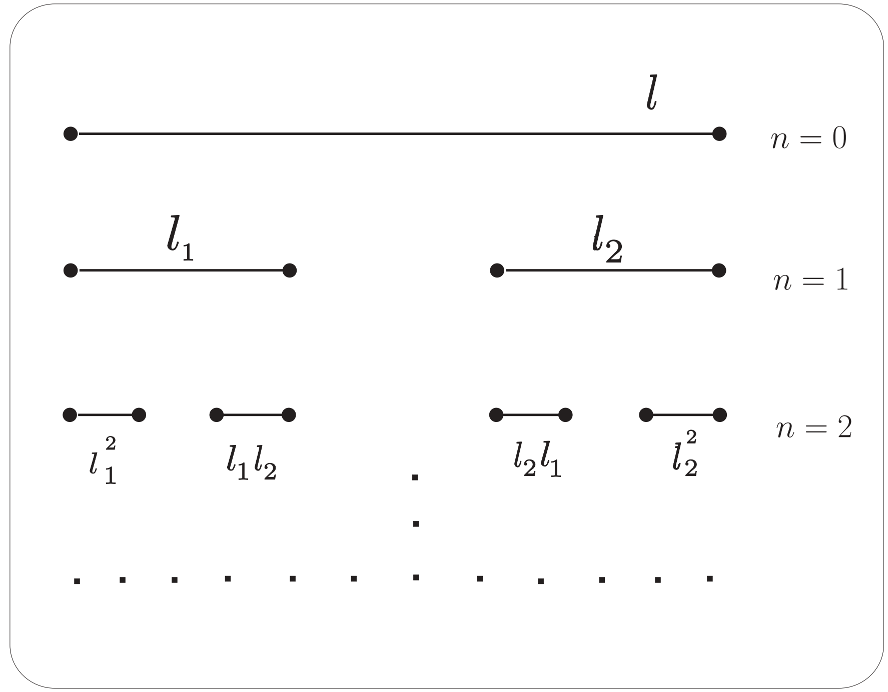
   - Fractal Analysis Basics
   - Importance of Multifractality
2. Solar Wind Data
   - Solar Wind Fluctuations
   - Space Missions (Helios, Ulysses, ACE/WIND)
3. Methods for Turbulence Models
   - Structure Functions Scaling
   - Energy Transfer Rate
   - Generalized Two-Scale Weighted Cantor Set
4. Results and Discussion
   - Generalized Two-Scale Mutifractal Model for Intermittent Turbulence
   - Comparison with the usual p-model (Meneveau and Sreenivasan, 1987)
   - Attractor Reconstruction
   - Dimensions and Multifractality
5. Conclusions

Prologue

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally *self-similar* and independent of scale (fractal dimension).

A **multifractal** is a set of intertwined fractals. Self-similarity of multifractals is scale dependent (spectrum of dimensions).



Two-scale **Cantor** set.

The question of multifractality is of great importance because it allows us to look at intermittent turbulence in the solar wind (e.g., Marsch and Tu, 1997; Bruno *et al.*, 2001). Starting from Richardson's scenario of turbulence, many authors try to recover the observed scaling exponents, using some simple and more advanced fractal and multifractal models of turbulence describing distribution of the energy flux between cascading eddies at various scales. In particular, the multifractal spectrum has been investigated using Voyager (magnetic field) data in the outer heliosphere (e.g., Burlaga, 1991, 2001) and using Helios (plasma) data in the inner heliosphere (e.g., Marsch *et al.*, 1996). The multifractal scaling has also been tested using Ulysses observations (Horbury et al., 1997) and with ACE/WIND data (e.g., Hnat *et al.*, 2003, 2007; Chapman *et al.*, 2006; Kiyani *et al.*, 2007).

A direct determination of the multifractal spectrum from the data is known to be a difficult problem. Indication for a chaotic attractor in the slow solar wind has been given by Macek (1998) and Macek and Redaelli (2000). In particular, Macek (1998) has calculated the correlation dimension of the reconstructed attractor in the solar wind and has provided tests for this measure of *complexity* including statistical surrogate data tests (Theiler *et al.*, 1992). Further, Macek and Redaelli (2000) have shown that the Kolmogorov entropy of the attractor is *positive* and finite, as it holds for a *chaotic* system.

We have extended our previous results on the dimensional time series analysis (Macek, 1998). Namely, we have applied the technique that allows a realistic calculation of the generalized dimensions of the solar wind flow directly from the cleaned experimental signal by using the Grassberger and Procaccia method. The resulting spectrum of dimensions shows the multifractal structure of the solar wind in the inner heliosphere (Macek *et al.*, 2005, 2006). Using a short data sample, we first demonstrate the influence of noise on these results and show that noise can efficiently be reduced by a singular-value decomposition filter (Macek, 2003, 2003). Using a longer sample we have shown that the multifractal spectrum of the solar wind attractor reconstructed in the phase space is consistent with that for the multifractal measure on the self-similar weighted baker's map (Macek *et al.*, 2005; Macek, 2006) and, in particular, with the weighted Cantor set (Macek *et al.*, 2006).

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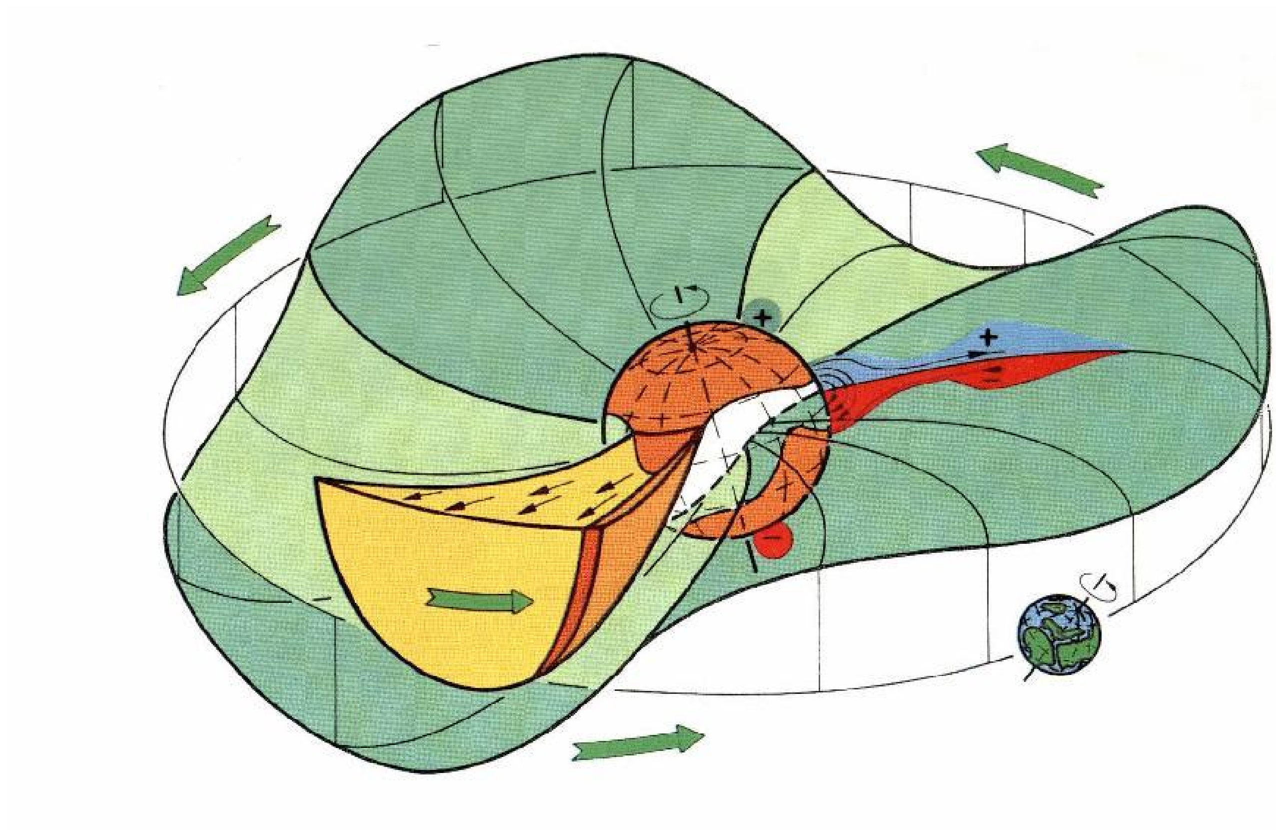
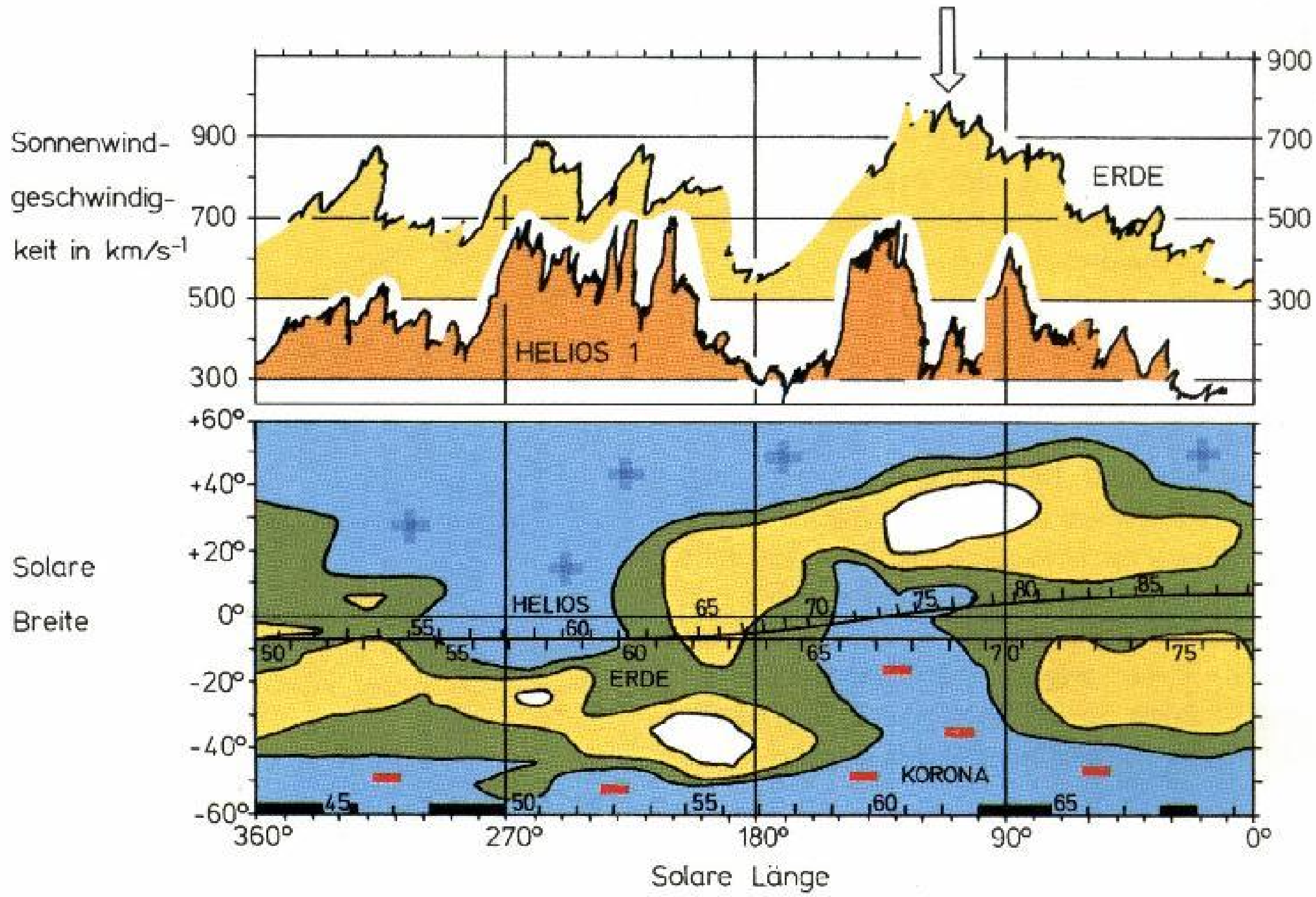
Here in order to further quantify that multifractality, we also consider a generalized weighted Cantor set with two different scales describing nonuniform compression (Macek, 2007). We investigate the resulting multifractal spectrum depending on two scaling parameters and one probability measure parameter, especially for asymmetric scaling. We hope that this model will be a useful tool for analysis of intermittent turbulence in space plasmas. In particular, taking two different scales for eddies in the cascade, one obtains a more general situation than in the usual p-model for fully developed turbulence (Macek and Szczepaniak, 2007).

Thus our results provide direct supporting evidence that the *complex* solar wind is likely to have multifractal structure. In this way, we have further supported our previous conjecture that trajectories describing the system in the inertial manifold of phase space asymptotically approach the attractor of low-dimension. One can expect that the attractor in the low-speed solar wind plasma should contain information about the dynamic variations of the coronal streamers. It is also possible that it represents a structure of the time sequence of near-Sun coronal fine-stream tubes (see, Macek, 1998, 2006, 2007), and references therein.



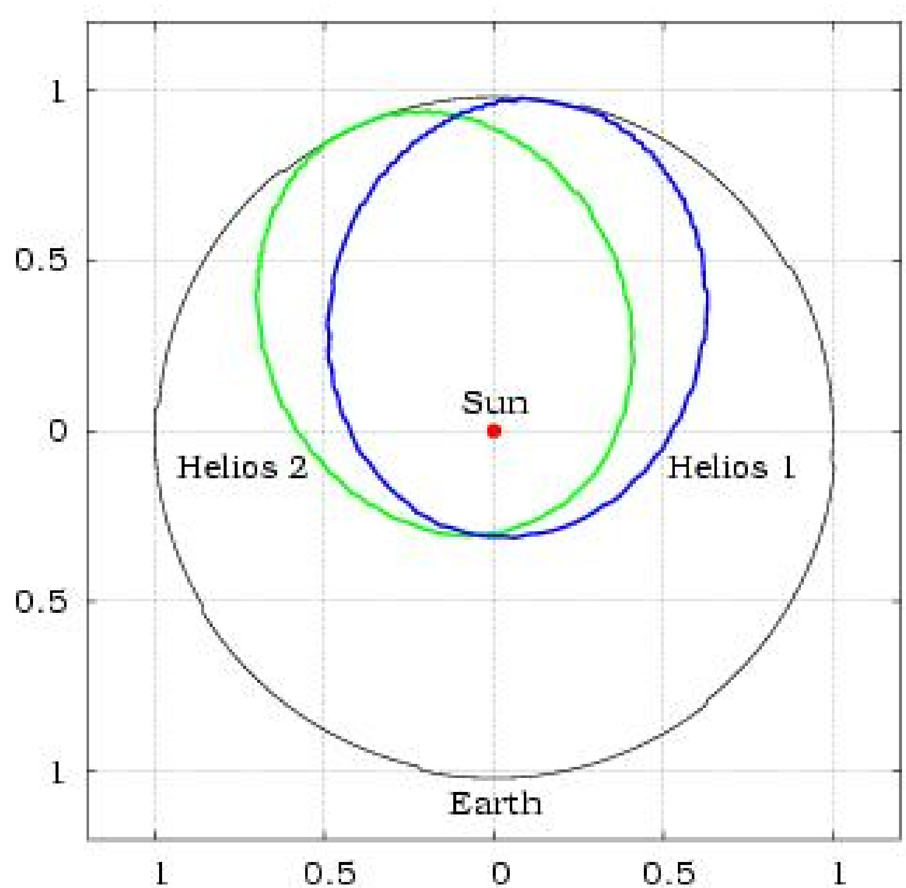
A schematic model of the solar wind "ballerina": the Sun's two hemispheres are separated by a neutral layer of a form reminiscent of a 'ballerina's skirt'. In the inner heliosphere the solar wind streams are of two forms called the slow (≈ 400 km s^{-1}) and fast (≈ 700 km s^{-1}). The fast wind is associated with coronal holes and is relatively uniform and stable, while the slow wind is quite variable, taken from (Schwenn and Rosenbauer, 1984).



Structures in the solar wind and their sources in the corona (solar map), taken from (Schwenn and Rosenbauer, 1984).

Helios Spacecraft

Orbits



Methods of Data Analysis

Structure Functions Scaling

$S_u^q(l)$, qth order structure function ($q > 0$) in the inertial range ($\eta \ll l \ll L$)

$$S_u^q(l) = \langle |u(x+l) - u(l)|^q \rangle \sim l^{\xi(q)} \tag{1}$$

$u(x)$, a velocity component parallel to the l
$\xi(q)$, a scaling exponent.

Energy Transfer Rate

$$\varepsilon_l \sim \frac{S_u^3(l)}{l} \qquad \mu_i = \frac{\varepsilon_l}{\langle \varepsilon_L \rangle} \tag{2}$$

$$\sum_i \mu_i^q \sim l^{\tau(q)} \tag{3}$$

$$\tau(q) = (q-1)\, D_q \tag{4}$$

ε_l, energy transfer rate
μ_i, probability measure of ith eddy in the d-dimensional physical space.

From Equations (1) to (4) we have (Tsang *et al.*, 2005):

$$D_q = d + \frac{\xi(3q) - q\xi(3)}{q-1} \tag{5}$$

Generalized Dimensions

The generalized dimensions are important characteristics of *complex* dynamical systems. Since these dimensions are related to frequencies with which typical orbits in phase space visit different regions of the system, they can provide information about its dynamics.

More precisely, one may distinguish a probability measure from its geometrical support, which may or may not have fractal geometry. Then, if the measure has different fractal dimensions on different parts of the support, the measure is multifractal. The modern technique of nonlinear time series analysis allows to estimate the multifractal measure directly from a single time series.

Mutifractal Models for Turbulence

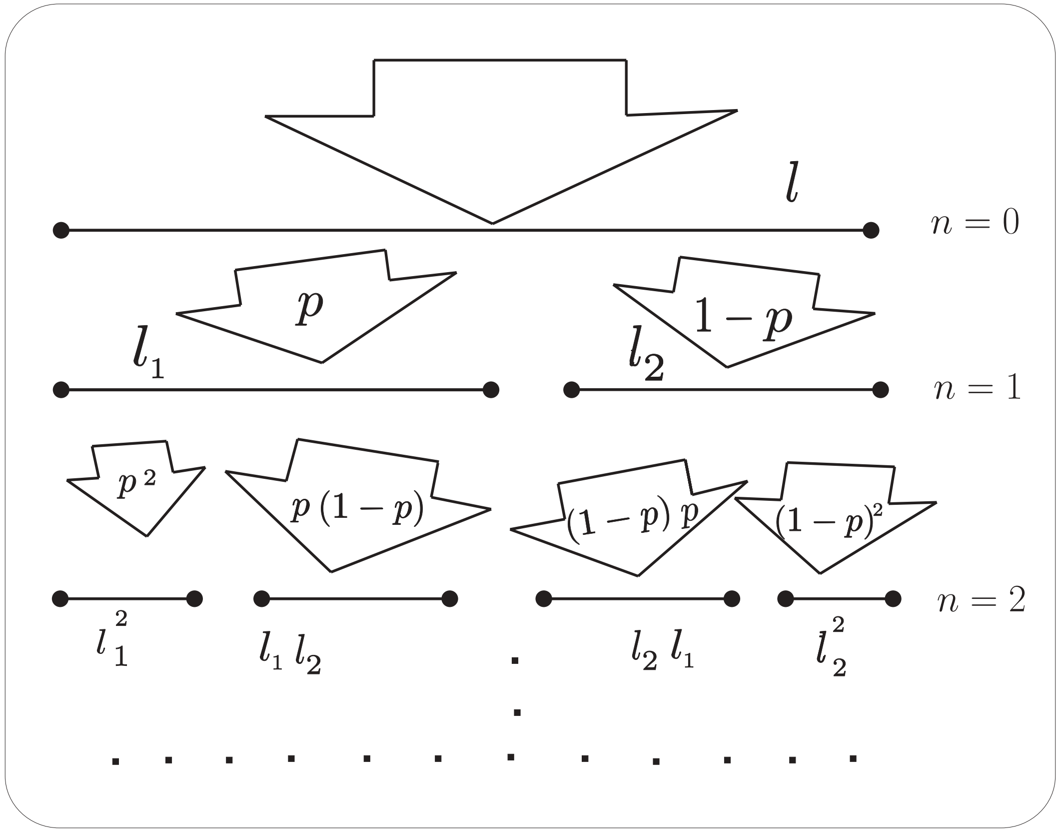


Fig. 1. Generalized two-scale weighted Cantor set model for fully develped turbulence.

For the generalized self-similar weighted Cantor set (acting on the unit interval) we use the following partition function at n-th level of construction (Hentschel and Procaccia 1983; Halsey *et al.*, 1986)

$$\Gamma_n(l_1, l_2, p_1, p_2) = \left(\frac{p_1^q}{l_1^{\tau(q)}} + \frac{p_2^q}{l_2^{\tau(q)}} \right)^n = 1 \tag{6}$$

Parameters:

- $p_1 = p \leq 1/2$, natural invariant measure on the attractor of the system, the probability of visiting one region of the interval (the probability of visiting the remaining region is $p_2 = 1 - p$);
- $l_1 + l_2 \leq 1$, two nonuniform compression (dissipation) parameters (stretching and folding in the phase space).

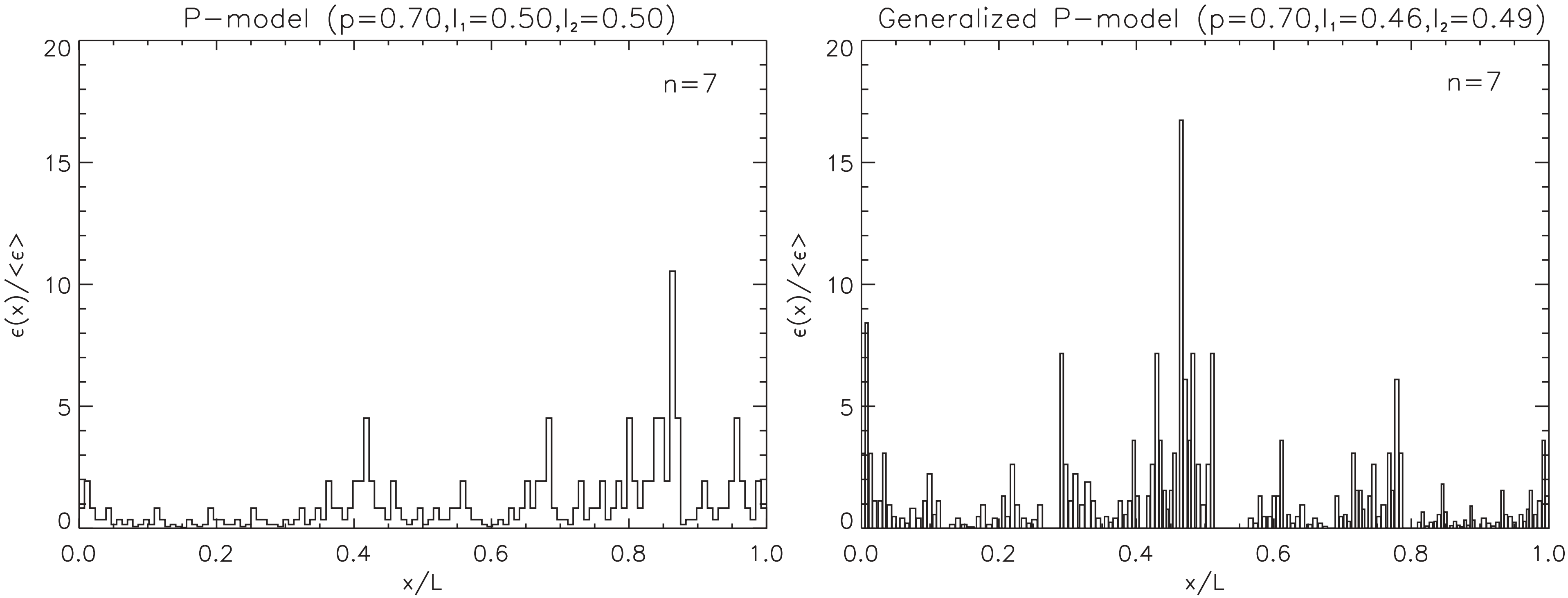


Fig. 2. The multifractal measure $\varepsilon/\langle\varepsilon\rangle$ on the unit interval for (a) the usual one-scale p-model (Meneveau and Sreenivasan, 1987) and (b) the generalized two-scale cascade model. Intermittent pulses are stronger for the model with two different scaling parameters (Macek and Szczepaniak, 2007).

Solutions

Transcendental equation (for $n \to \infty$)

$$p^q \, l_1^{-\tau(q)} + (1-p)^q \, l_2^{-\tau(q)} = 1 \qquad (7)$$

Legendre transformation

$$\alpha(q) = \frac{\mathrm{d}\ \tau(q)}{\mathrm{d}q} \qquad (8)$$

$$f(\alpha) = q\alpha(q) - \tau(q) \qquad (9)$$

For $l_1 = l_2 = s$ and any q in Eq. (7) one has for the generalized dimension of the attractor (projected onto one axis)

$$(q-1)D_q = \frac{\ln[p^q + (1-p)^q]}{\ln s}. \tag{10}$$

No dissipation ($s = 1/2$):
the multifractal cascade $p-$model for fully developed turbulence,
the generalized weighted Cantor set (Meneveau and Sreenivasan, 1987).
The usual middle one-third Cantor set (without any multifractality):
$p = 1/2$ and $s = 1/3$.

The difference of the maximum and minimum dimension
(the least dense and most dense points on the attractor)

$$D_{-\infty} - D_{+\infty} = \frac{\ln(1/p-1)}{\ln\,(1/s)} \tag{11}$$

In the limit $p \to 0$ this difference rises to infinity (degree of multifractality).

Data

Table 1: The time intervals of Helios 2 data in 1976 for slow and fast solar wind streams measured at various distances from the Sun.

	~ 0.3 AU	~ 0.97 AU
Slow streams	099:00:00:29 - 099:23:34:27	026:00:00:33 - 026:23:58:45
	100:00:09:57 - 100:23:59:41	027:00:00:05 - 027:23:59:57
ddd:hh:mm:ss	101:00:00:21 - 101:23:59:47	028:00:00:39 - 028:23:59:59
	102:00:00:27 - 102:23:59:31	029:00:00:41 - 029:23:59:23
Fast streams	105:00:00:31 - 105:23:59:25	021:00:00:09 - 021:23:59:41
	106:00:00:45 - 106:23:59:35	022:00:00:17 - 022:23:59:51
ddd:hh:mm:ss	107:00:00:17 - 107:23:59:49	023:00:00:31 - 023:23:59:35
	108:00:00:29 - 108:23:59:55	024:00:00:07 - 024:23:59:43

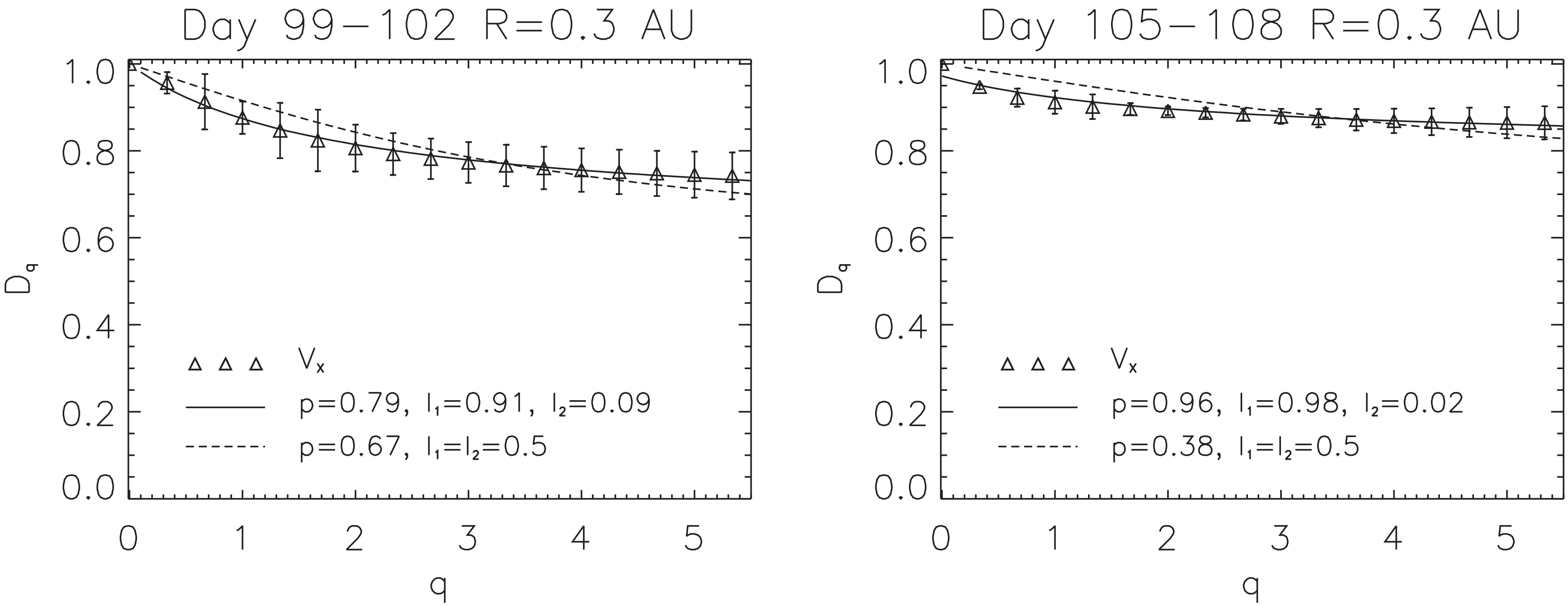
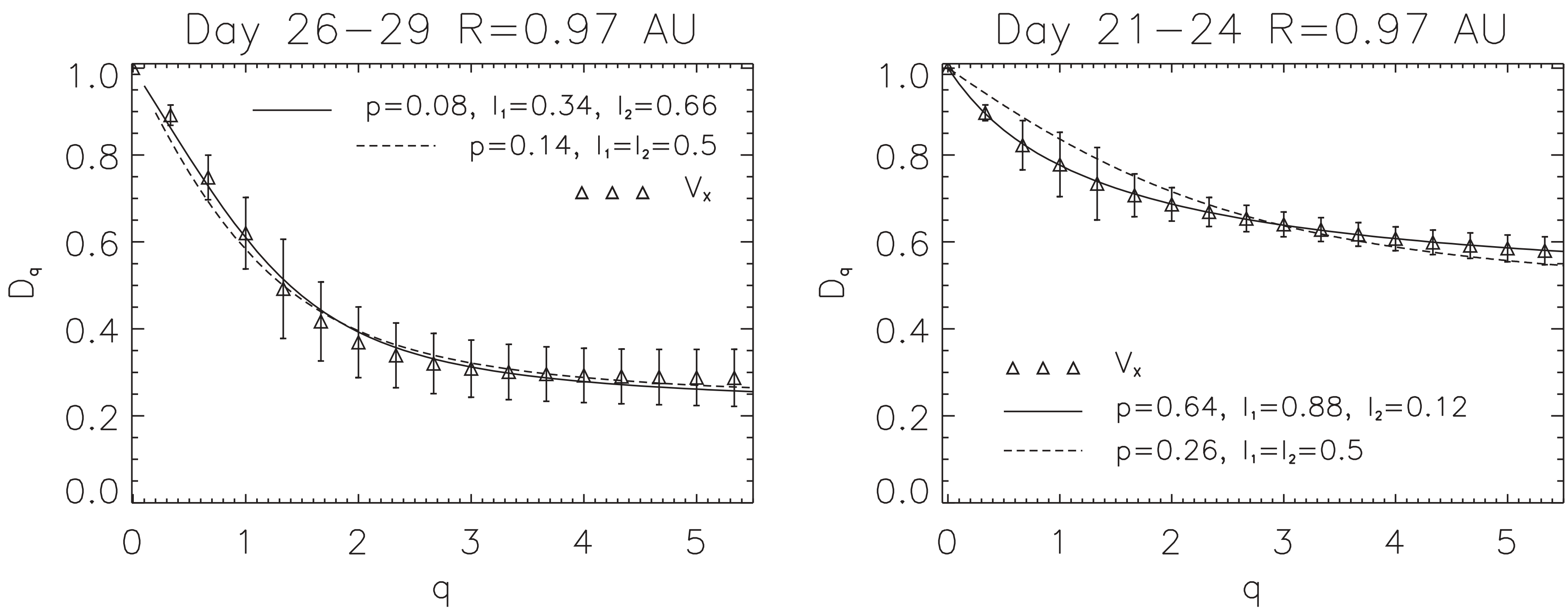


Fig. 3. The generalized dimensions D_q as a function of q. The values of D_q are calculated for the generalized two-scale (continuous lines) model and the usual one-scale (dashed lines) p-model and fitted using the V_x velocity components (triangles) for the slow (a) and fast (b) solar wind streams at distances of 0.3 AU (Macek and Szczepaniak, 2007).



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Attractor Reconstruction

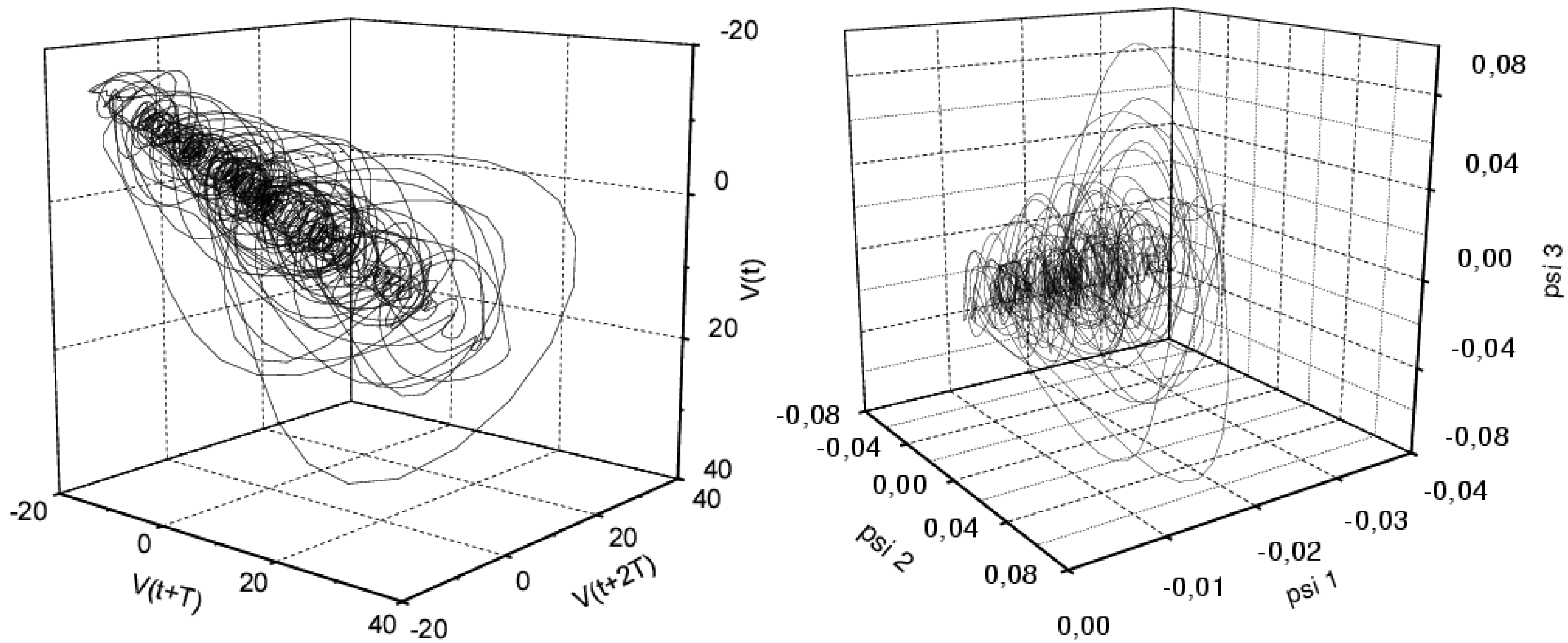

Fig. 4. The projection of the attractor onto the three-dimensional space, reconstructed from the detrended data, $T = 4\,\Delta t$, using (a) the moving average and also (b) the singular-value decomposition filters ($\Psi = U$), taken from (Macek, 1998).

Alfvénic Velocity

Sound velocity:	$c_s^2 = \gamma \frac{p}{\rho}$
Magnetic field pressure:	$p = \frac{B^2}{8\pi}$
Adiabatic exponent:	$\gamma = \frac{f+2}{f} = 2 \quad f = 2$
Alfvénic velocity:	$v_A = \frac{B}{\sqrt{4\pi\rho}}$

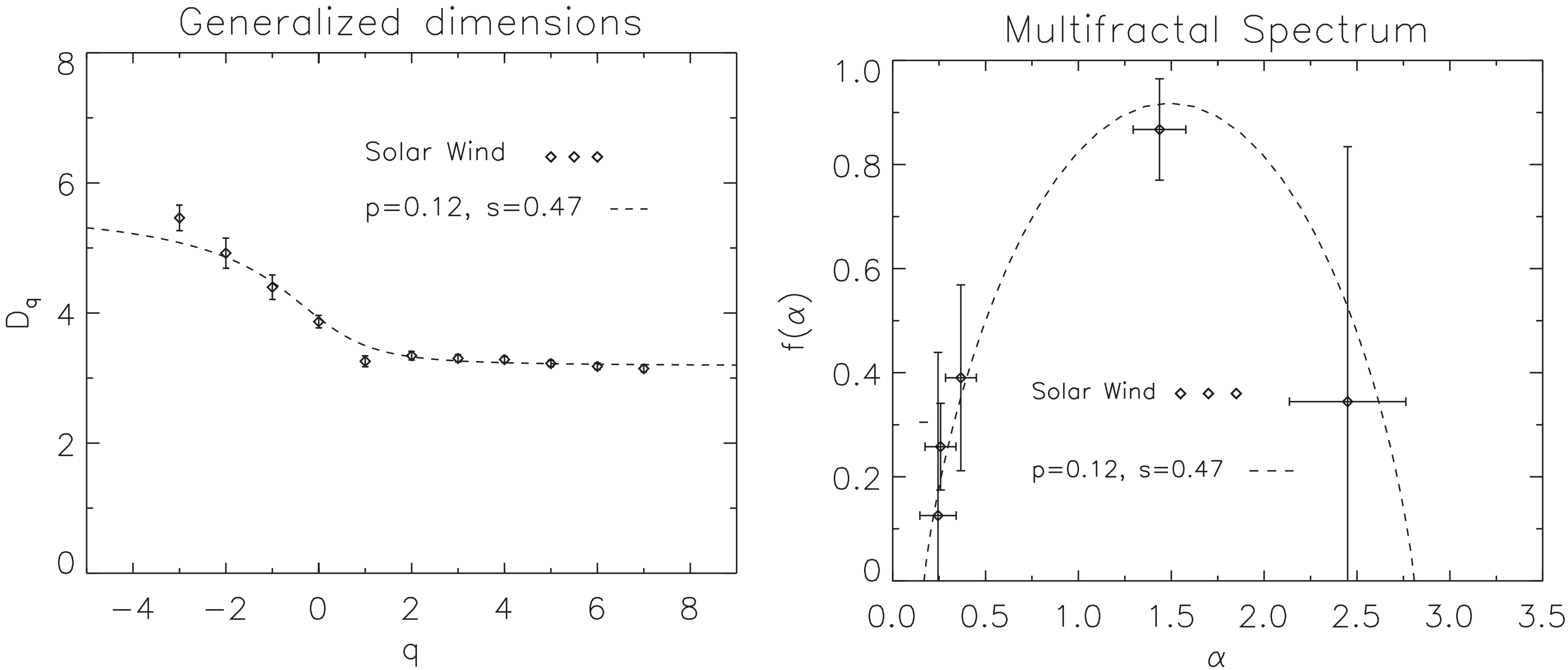

Fig. 5. (a) The generalized dimensions D_q in Equation (4) as a function of q. The correlation dimension is $D_2 = 3.4 \pm 0.1$. The values of $D_q + 3$ are calculated analytically for one-scale weighted Cantor set (baker's map) with $p = 0.12$ and $s = 0.47$ (dashed line). (b) The singularity spectrum $f(\alpha)$ as a function of α. The values of $f(\alpha)$ projected onto one axis for the weighted baker's map with the same parameters (dashed line), taken from (Macek, 2006).

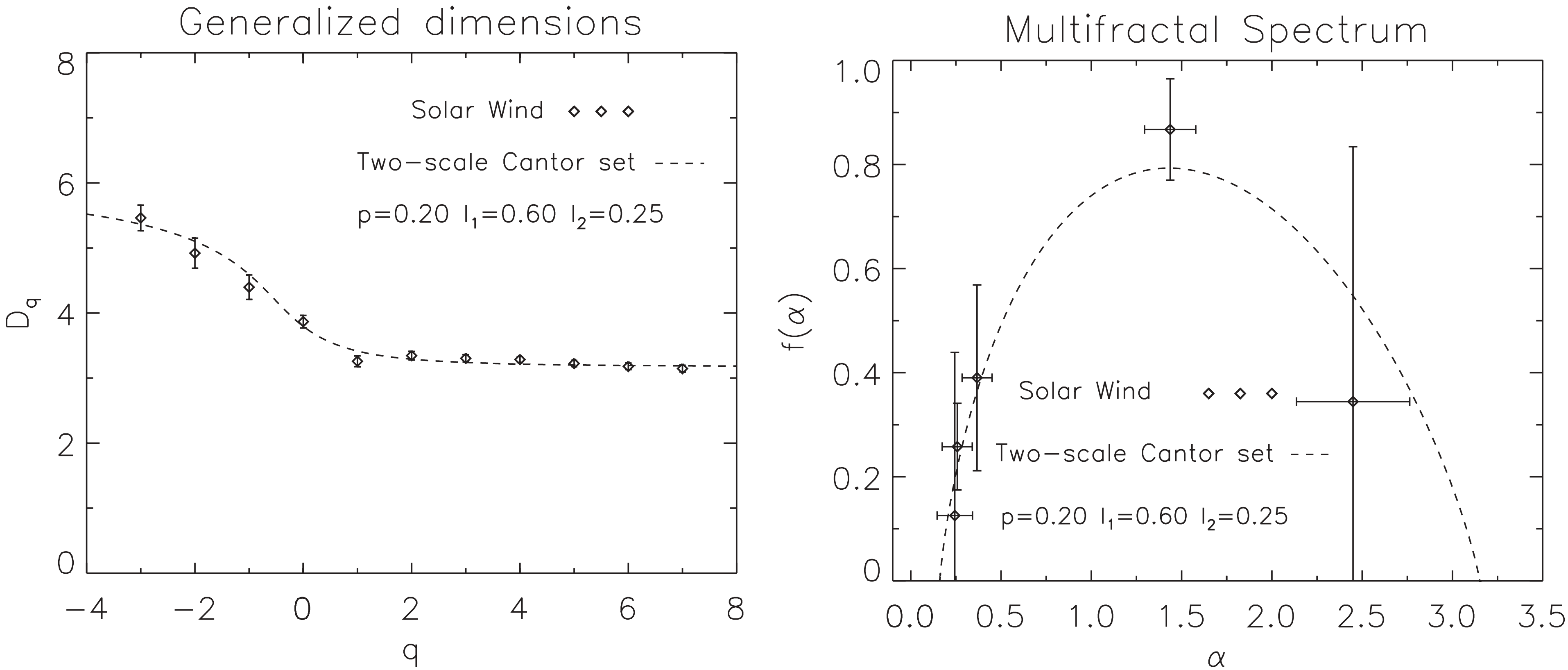

Fig. 6. (a) The generalized dimensions D_q in Equation (4) as a function of q. The values of $D_q + 3$ are calculated analytically for the weighted two-scale Cantor set with $p = 0.20$ and $l_1 = 0.60$, $l_2 = 0.25$ (dashed line). (b) The singularity spectrum $f(\alpha)$ as a function of α. The values of $f(\alpha)$ projected onto one axis for the weighted two-scale Cantor set with the same parameters (dashed line), taken from (Macek, 2007).

The value of parameter p (within some factor) is related to the usual models, which starting from Richardson's scenario of turbulence, try to recover the observed scaling exponents, which is based on the p-model of turbulence (e.g. Meneveau and Sreenivasan, 1987).

The value of $p \approx 0.2$ obtained here is roughly consistent with the fitted value in the literature both for laboratory and the solar wind turbulence, which is in the range $0.13 \leq p \leq 0.3$ (e.g., Burlaga, 1991; Carbone, 1993; Carbone and Bruno, 1996; Marsch *et al.*, 1996).

One should only bear in mind that here we take probability measure directly on the solar wind attractor, which quantifies multifractal nonuniformity of visiting various parts of the attractor in the phase space, while the usual p-model is related to the solar wind turbulence cascade for the energy transfer rate, which resides in the physical space.

Conclusions

- We have studied departure from Kolmogorov scaling indicating multifractal (intermittent) behavior of the solar wind in the inner heliosphere.
  - We confirm that the degree of multifractality of the solar wind is different for slow and fast streams.
  - Our analysis shows that slow solar wind velocity fluctuations are more intermittent and more anisotropic than for the fast solar wind.
  - Also as the heliocentric distance increases the solar wind becomes more multifractal consistent with other studies. In particular, we observe radial evolution of multifractality (intermittency) as noticed, e.g., by Marsch *et al.* (1996).
- Basically, the generalized dimensions for solar wind are not only consistent with the standard p-model (Meneveau and Sreenivasan, 1987), but rather with the more general model with two different scaling parameters for sizes of eddies. In particular, we show that intermittent pulses are stronger for the model with asymmetric scaling and a somewhat better agreement with the solar wind data is obtained.
- Therefore, we propose the generalized two-scale weighted Cantor set model describing intermittent energy transfer cascade for analysis of space plasma turbulence.

In this way, we have further supported our previous conjecture that

- trajectories describing the system in the inertial manifold of phase space asymptotically approach the attractor of low-dimension (Macek, 1998).
- The obtained multifractal spectrum of this attractor is consistent with that for the multifractal measure on the generalized weighted two-scale Cantor set, which is a strange attractor that exhibits stretching and folding properties leading to sensitive dependence on initial conditions (Macek, 2006, 2007).
- The values of the parameters fitted demonstrates small dissipation of the complex solar wind dynamical system and shows that some parts of the attractor in phase space are visited at least one order of magnitudes more frequently than other parts.

One can expect that the attractor in the low-speed solar wind plasma should contain information about the dynamic variations of the coronal streamers. It is also possible that it represents a structure of the time sequence of near-Sun coronal fine-stream tubes.

Epilogue

Thus these results provide supporting evidence for **multifractal** structure of the solar wind in the inner heliosphere.

This means that the observed **intermittent** behavior of the solar wind's velocity and Alfvénic fluctuations results from intrinsic *nonlinear* dynamics rather than from random external forces.

The multifractal structures, convected by the solar wind, might probably be related to the complex topology shown by the magnetic field at the source regions of the solar wind.

Thank you!

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