Weighted two-scale Cantor set in the solar wind turbulence

A. Szczepaniak * (1) , W.M. Macek (1,2)

(1) Space Research Centre, Polish Academy of Science, Bartycka 18A, 00-716 Warsaw, Poland

(2) Faculty of Mathematics and Natural Sciences. College of Sciences, Cardinal Stefan Wyszyński University, Dewajtis 5, 01-815 Warsaw, Poland

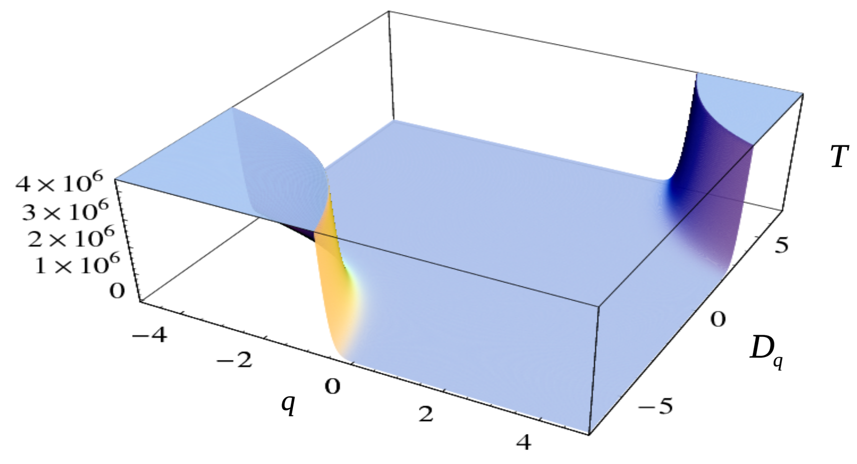
*Email: sanna@cbk.waw.pl

ABSTRACT

Multifractality helps us to describe fluctuations that are non-Gaussian, bursty, nonhomogeneous, in particular, in systems related to the intermittent turbulence of the solar wind [2-5]. Using generalized dimensions and singularity spectra we can better understanding dynamic of multifractal time series. Here we analyse the time series of the solar wind plasma parameters measured in situ by Helios 2 spacecraft in the inner heliosphere. We use Grassberger and Procaccia method for calculation of the generalized dimensions of the solar wind attractor in the phase space directly from the cleaned experimental signal. The obtained solar wind spectrum is consistent with that for the multifractal measure on the weighted baker's map [5,6]. We also analyse a theoretical model for generating multifractals, i.e. weighted two-scale Cantor set [1]. This example of multifractal has two scaling parameters (l_1, l_2) and a probability measure (p). We solve numerically transcendental equation in order to obtain generalized dimensions and singularity spectrum for the two-scale Cantor set. We analyse degree of multifractality of this theoretical model depending on the parameter p. Finally, we compare the resulting generalized dimensions for the solar wind and for the weighted two-scale Cantor set. We consider similarities and differences between resulting spectra. This comparison shows similar character of multifractality of the solar wind and the two-scale Cantor set.

METHODS

- **Transcendental equation**



$$T(q, D_q) = p^q l_1^{(1-q)D_q} + (1-p)^q l_2^{(1-q)D_q} - 1$$

The shape of the T as a function of q and D_q for $p = 0.20,\ l_1 = 0.60,\ l_2 = 0.25$.

Direct relation between q and $\tau(q)$ for the weighted two-scale Cantor set is obtained from the following transcendental equation:

$$p^q l_1^{-\tau(q)} + (1-p)^q l_2^{-\tau(q)} = 1 \qquad (2)$$

When $l_1 = l_2 = s$, equation (2) can be solved analyticaly and one obtain the generalized dimensions for the weighted one-scale Cantor set.

$$\tau(q) \equiv (1-q)D_q = \frac{\ln[p^q + (1-p)^q]}{\ln s}$$

- **Grassberger and Procaccia algorithm**

To obtain generalized dimensions (1) we use following relations:

$$p_j \approx \frac{1}{n - 2n_c - 1} \sum_{i=n_c+1}^{n} \theta(e - |\mathbf{X}(t_i) - \mathbf{X}(t_j)|)$$

where $\theta(x)$ is the unit step function,

$\mathbf{X}(t_i) = [x(t_i), x(t_i + \tau), ..., x(t_i + (m-1)\tau)]$,

vectors in the embedding phase space of dimension m,

$n = N - (m-1)\tau$, $n_c = 4$ - Theiler's correction.

We take into account q -point correlation sum defined by

$$C_q(m,e) = \frac{1}{n_{\text{ref}}} \sum_{j=1}^{n_{\text{ref}}} (p_j)^{q-1}$$

where $n_{\text{ref}} = 5000$ is the number of reference vectors. For large dimensions m and small e in the scaling region it can be argued that

$$C_q(m,e) \propto e^{\tau(q)}$$

INTRODUCTION

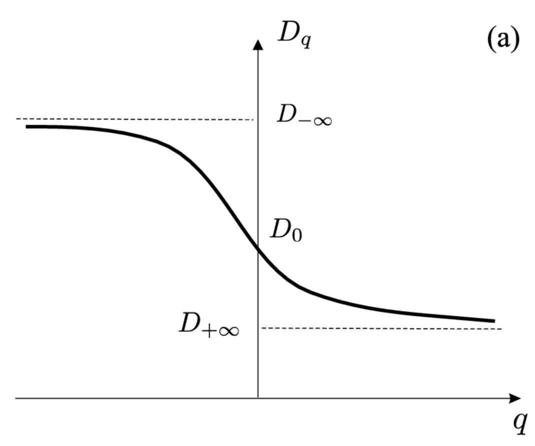
Theory of multifractals is related to the generalized dimensions. The q-order generalized dimension is given by

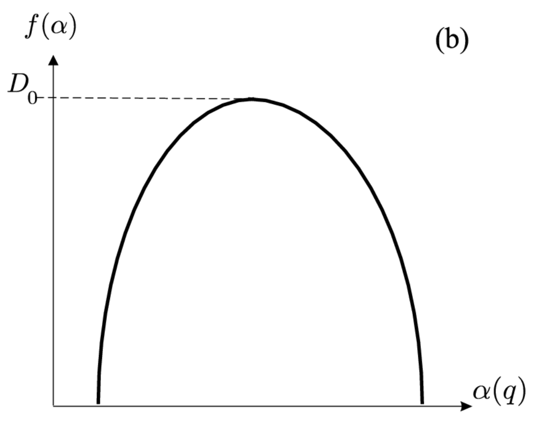
$$D_q = \frac{1}{1-q} \lim_{e\to\infty} \frac{\ln \sum_j (p_j)^q}{\ln(e)} \qquad (1)$$

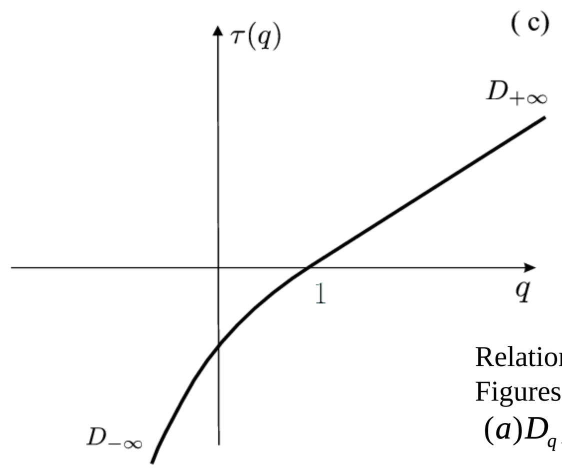
where p_j is the probability measure that a point from a time series falls in the hypercube of size e. An intuitive picture about the inner structure of a multifractal is obtained by introducing the multifractal spectrum, $f(\alpha)$. The $f(\alpha)$ is connected with $\tau(q) = (q-1)D_q$ by Legendre transformation:

$$\alpha(q) = \frac{d\tau}{dq}$$

$$f(\alpha) = q\alpha(q) - \tau(q)$$







Relation between D_q, $\tau(q)$ and $f(\alpha)$. Figures show general shape of the function $(a) D_q$, (b) $f(\alpha)$, (c) $\tau(q)$.

DATA

- **Weighted two-scale Cantor set**

Cantor set

$p = \frac{1}{2}, l_1 = l_2 = \frac{1}{3}$

Baker's map

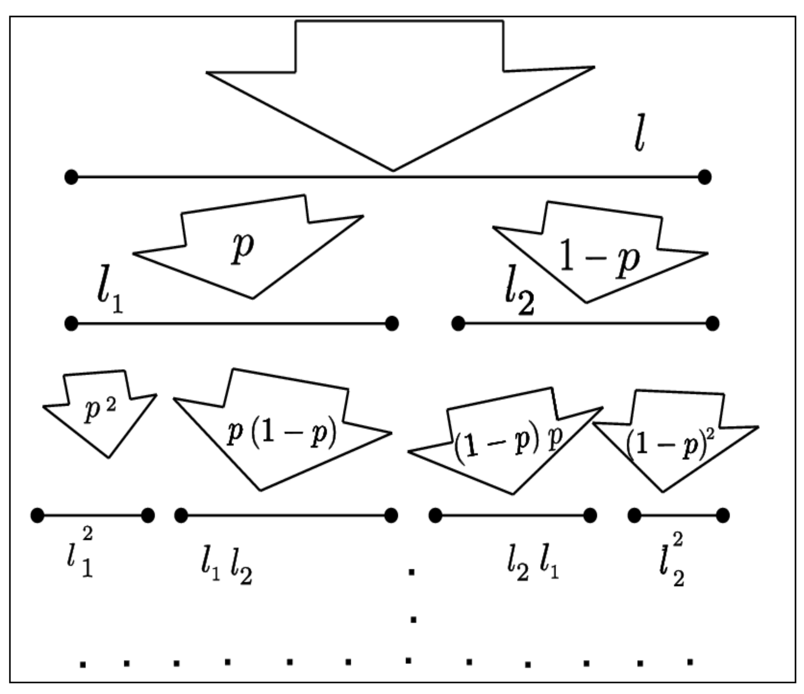
$p = l_1 = l_2 = \frac{1}{2}$

Weighted one-scale Cantor set

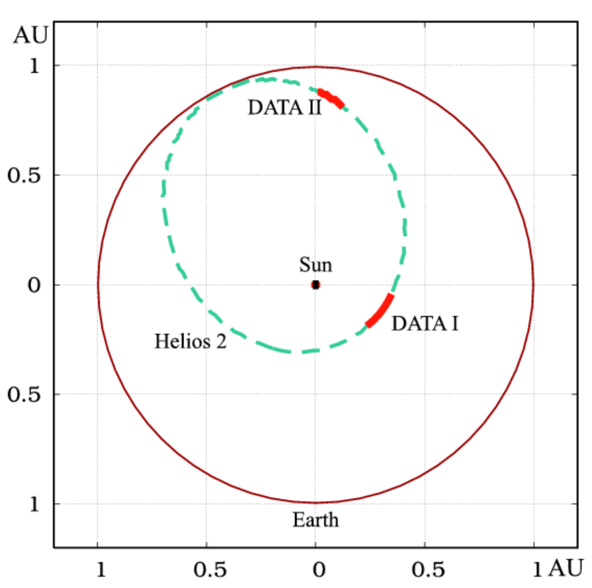
$l_1 = l_2 = s \le \frac{1}{2}$

Weighted two-scale Cantor set

$l_1 \ne l_2$



- **Solar wind**

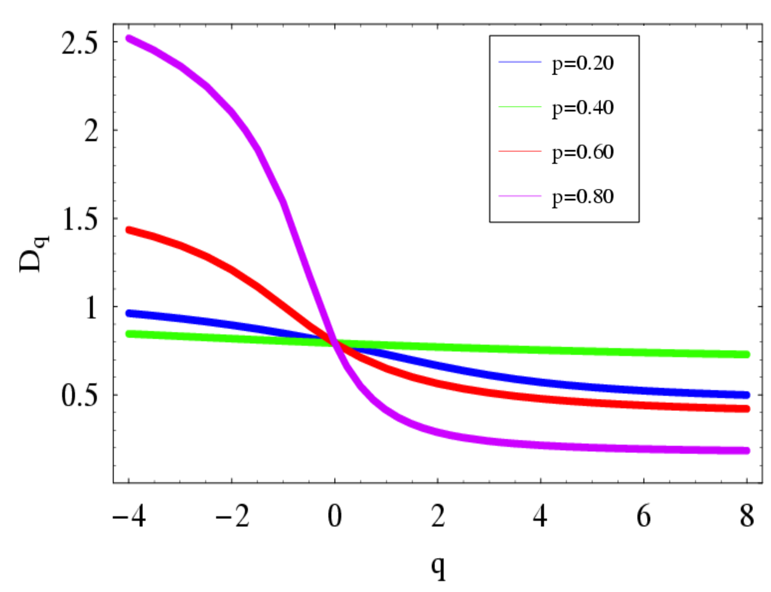


We analyse the Helios 2 data ($N = 26163$ points, $\Delta t = 40.5$s) using plasma parameters measured in situ in the inner heliosphere. The raw data of the radial flow velocity with Alfvénic velocity, v and $v_A = B/\sqrt{m_0 r}$ has been observed by the Helios 2 spacecraft in 1977 from 116:00 to 121:21 (day:hour) at distance 0.3 AU and from 348:00 to 357:00 at distance 0.9 AU from the Sun.

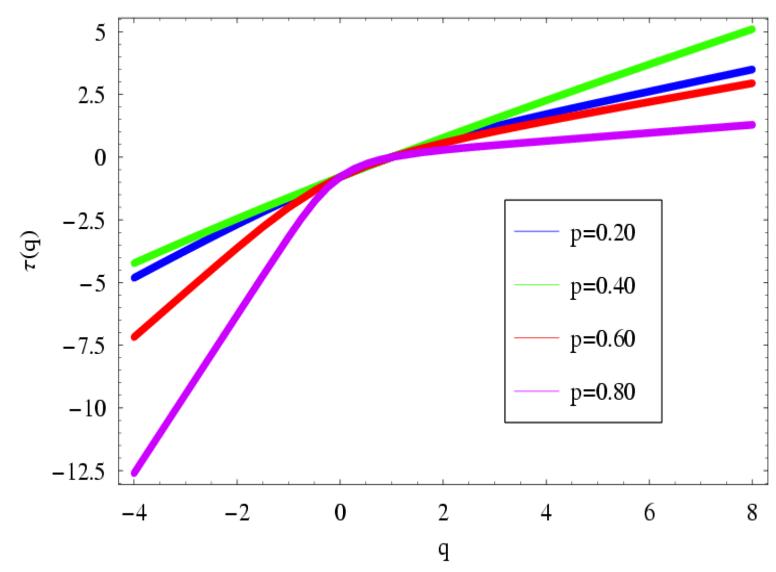
We take into account the radial component of the Elsässer variable $z_\pm = v \pm v_A$ for the unperturbed magnetic field B_0 pointing to/away from the Sun. The data were detrended and filtered using singular-value decomposition.

RESULTS

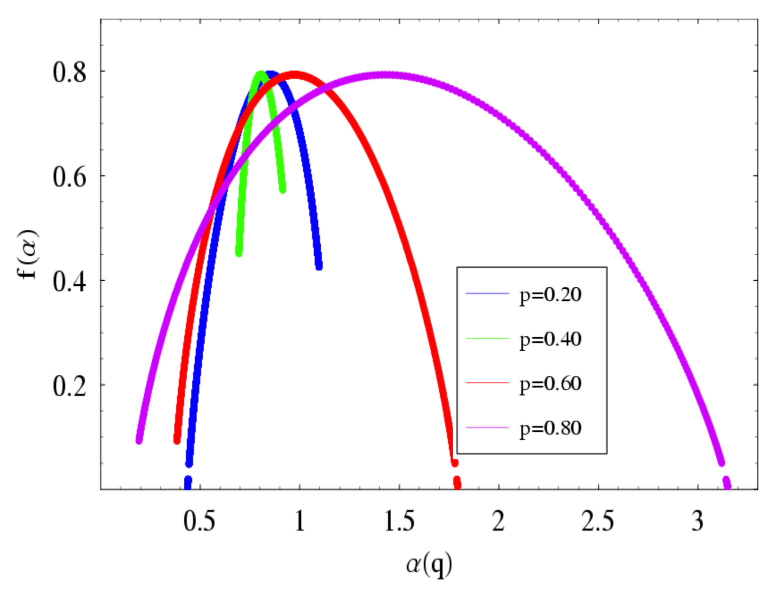
- **Weighted two-scale Cantor set**



The generalized dimensions as a function of q for the weighted two-scale Cantor set using different values of p $l_1 = 0.60,\ l_2 = 0.25$.

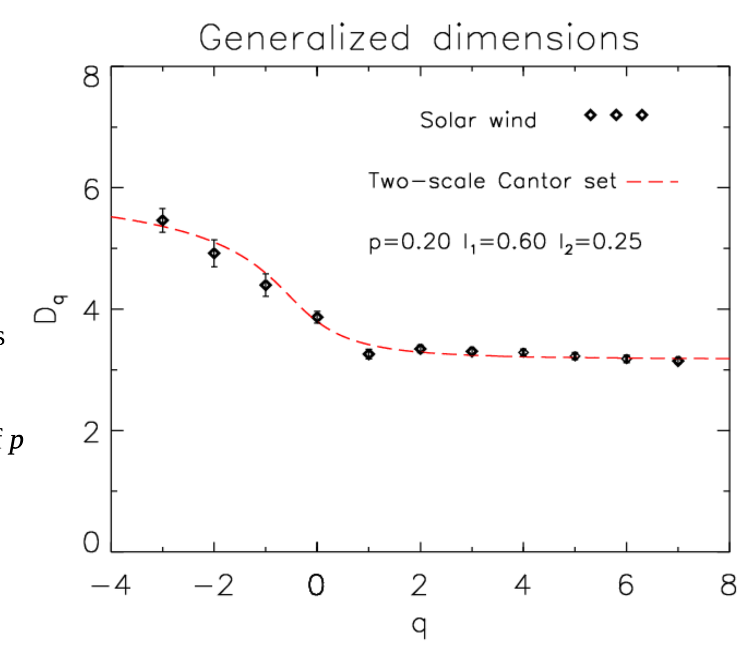


The $\tau(q)$ as a function of q for the weighted two-scale Cantor set using different value of p. $l_1 = 0.60,\ l_2 = 0.25$.

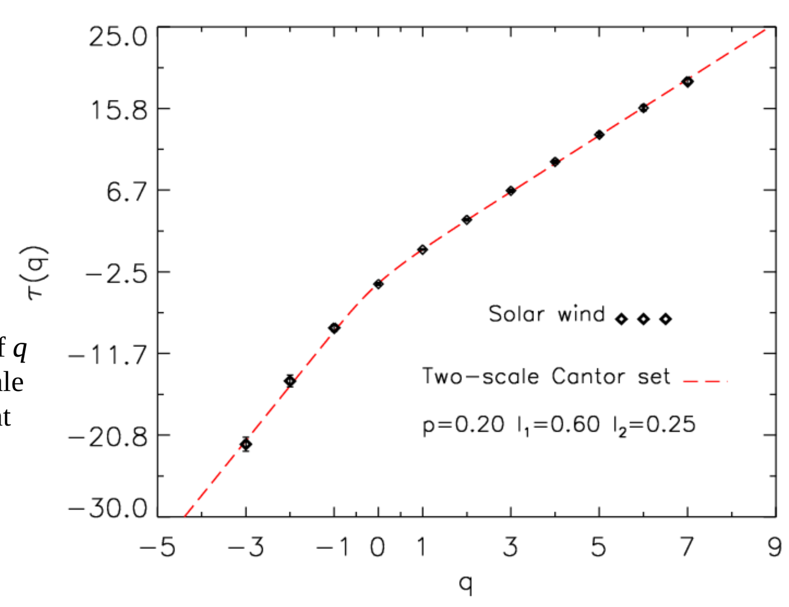


The multifractal spectrum $f(\alpha)$ as a function of $\alpha(q)$ for the weighted two-scale Cantor set using different values of p. $l_1 = 0.60,\ l_2 = 0.25$.

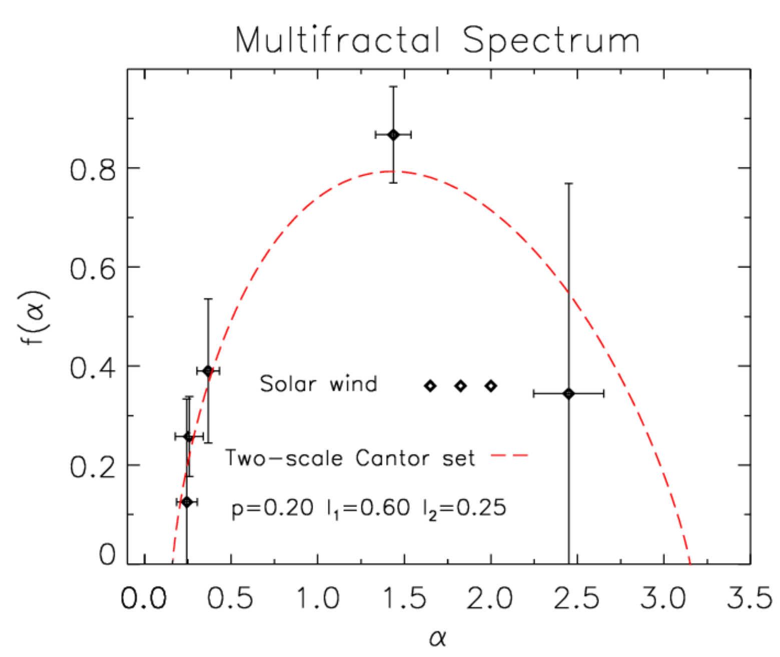
- **Comparison to the solar wind data**



The generalized dimensions D_q as a function of q. The values of $D_q + 3$ are calculated numerically for the weighted two-scale Cantor set using $p = 0.20,\ l_1 = 0.60,\ l_2 = 0.25$ (dashed-dotted) and fitted to the solar wind (diamonds).



The $\tau(q)$ as a function of q. The values of $\tau(q) + 3$ are calculated numerically for the weighted two-scale Cantor set using $p = 0.20,\ l_1 = 0.60,\ l_2 = 0.25$ (dashed-dotted) and fitted to the solar wind (diamonds).



The multifractal spectrum $f(\alpha)$ as a function of $\alpha(q)$ for the weighted two-scale Cantor set using $p = 0.20,\ l_1 = 0.60,\ l_2 = 0.25$ (dashed-dotted) and fitted to the solar wind (diamonds).

CONCLUSIONS

- We have shown that degree of multifractality of the weighted two-scale Cantor set is sensitive to parameter p.
- Generalized dimensions for solar wind are consistent with the weighted two-scale Cantor set with $p = 0.20,\ l_1 = 0.60,\ l_2 = 0.25$ or $p = 0.80,\ l_1 = 0.25,\ l_2 = 0.60$.
- Negative values of q are less reliable (basic statistical problems)
- We propose to use weighted two-scale Cantor set for solar wind turbulence as a model of intermittent dissipation energy cascade.

REFERENCES

[1] T.C. Halsey, M.H. Jensen, L.P. Kadano., I. Procaccia, B.I.Shraiman, "Fractal measures and their singularities: The characterization of strange sets", *Phys. Rev.* A, **33**, 1141 (1986).

[2] C. Meneveau, K.R. Sreenivasan, "Simple multifractal cascade model for fully developed turbulence", *Phys. Rev. Lett.* **59**, 1424 (1987).

[3] L.F. Burlaga, "Multifractal structure of the interplanetary magnetic field: Voyager 2 observations near 25 AU, 1987-1988", *Geophys. Res. Lett.* **18**, 69 (1991).

[4] E. Marsch, C.-Y. Tu, H. Rosenbauer, " Multifractal scaling of the kinetic energy flux in solar wind turbulence", *Ann. Geophys.* **14**, 259 (1996).

[5] W.M. Macek, R. Bruno, G. Consolini, "Generalized dimensions for fluctuations in the solar wind", *Phys. Rev.* E, **72**, 017202 (2005).

[6] W.M. Macek, "Modeling multifractality of the solar wind ", Space Sci. Rev. 122, 329 (2006).