Fractals and Multifractals

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Objective

The aim of the course is to give students an introduction to the new developments in nonlinear dynamics and (multi-)fractals. Emphasis will be on the basic concepts of fractals, multifractals, chaos, and strange attractors based on intuition rather than mathematical proofs. The specific exercises will also include applications to intermittent turbulence in various environments. On successful completion of this course, students should understand and apply the fractal models to real systems and be able to evaluate the importance of multifractality with possible applications to physics, astrophysics and space physics, chemistry, biology, and even economy.

Plan of the Course

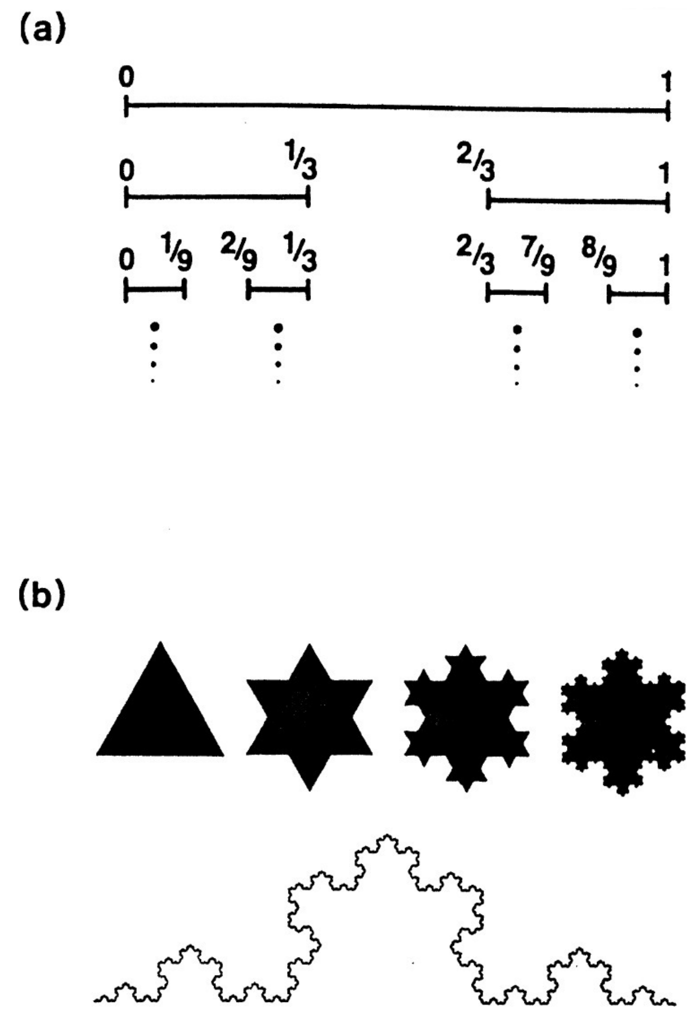
1. Introduction
   - Dynamical and Geometrical View of the World
   - Fractals
   - Attracting and Stable Fixed Points
2. Fractals and Chaos
   - Deterministic Chaos
   - Strange Attractors
3. Multifractals
   - Intermittent Turbulence
   - Weighted Two-Scale Cantors Set
   - Multifractals Analysis of Turbulence
4. Applications and Conclusions

- Importance of Being Nonlinear
- Importance of Multifractality

Fractals

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole.

Fractals are generally *self-similar* and independent of scale (fractal dimension).



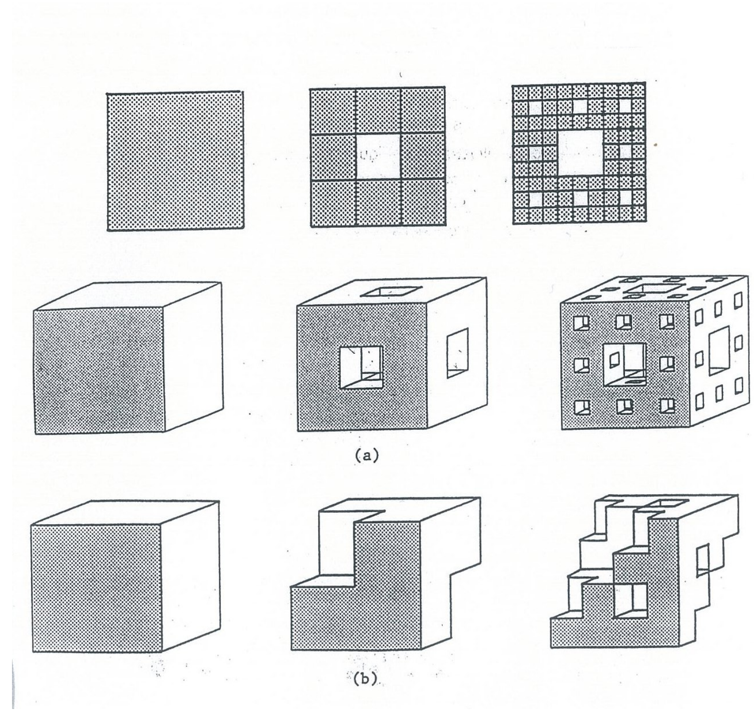
If N_n is the number of elements of size r_n needed to cover a set (C is a constant) is:

$$N_n = \frac{C}{{r_n}^D}, \qquad (1)$$

then in case of self-similar sets:
$N_{n+1} = C/(r_{n+1})^D$,
and hence the fractal similarity dimension D is

$$D = \ln(N_{n+1}/N_n)/\ln(r_n/r_{n+1}). \qquad (2)$$

- Cantor set $D = \ln 2/\ln 3$
- Koch curve $D = \ln 4/\ln 3$
- Sierpinski carpet $D = \ln 8/\ln 3$
- Mengor sponge $D = \ln 20/\ln 3$
- Fractal cube $D = \ln 6/\ln 2$

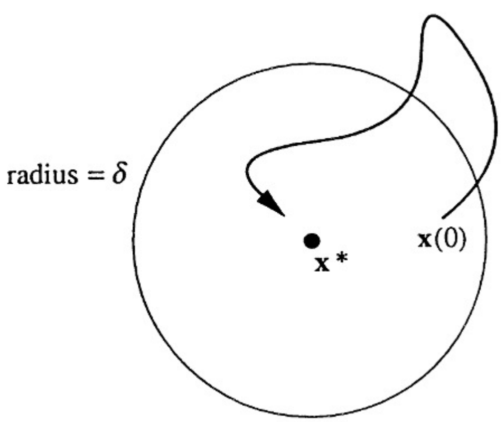


Attracting and Stable Fixed Points

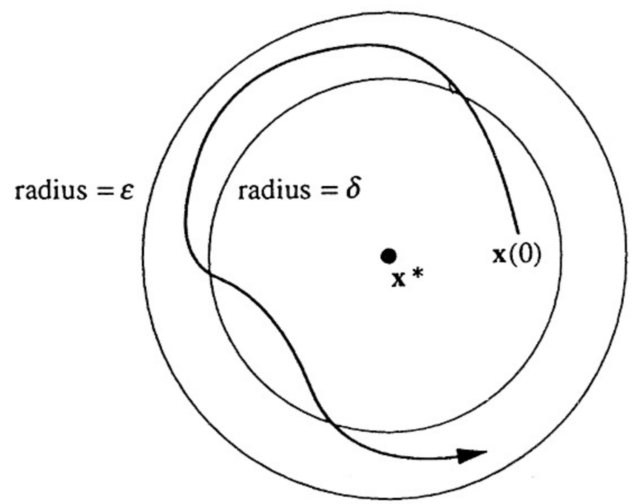
We consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, where $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$.

We say that $\mathbf{x}^*$ is *attracting* if there is a $\delta > 0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$ whenever $|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$: any trajectory that starts within a distance δ of $\mathbf{x}^*$ is guaranteed to converge to $\mathbf{x}^*$.

A fixed point $\mathbf{x}^*$ is *Lyapunov stable* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$ whenever $t \geq 0$ and $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$: all trajectories that start within δ of $\mathbf{x}^*$ remain within ε of $\mathbf{x}^*$ for all positive time.



Attracting



Liapunov stable

Deterministic Chaos

CHAOS ($\chi\alpha o\varsigma$) is

- NON-PERIODIC long-term behavior
- in a DETERMINISTIC system
- that exhibits SENSITIVITY TO INITIAL CONDITIONS.

We say that a bounded solution $\mathbf{x}(t)$ of a given dynamical system is SENSITIVE TO INITIAL CONDITIONS if there is a finite fixed distance $r > 0$ such that for any neighborhood $\|\Delta\mathbf{x}(0)\| < \delta$, where $\delta > 0$, there exists (at least one) other solution $\mathbf{x}(t) + \Delta\mathbf{x}(t)$ for which for some time $t \geq 0$ we have $\|\Delta\mathbf{x}(t)\| \geq r$.

There is a fixed distance r such that no matter how precisely one specify an initial state there is a nearby state (at least one) that gets a distance r away.

Given $\mathbf{x}(\mathbf{t}) = \{x_1(t), \ldots, x_n(t)\}$ any positive finite value of Lyapunov exponents $\lambda_k = \lim_{t\to\infty} \frac{1}{t} \ln \left| \frac{\Delta x_k(t)}{\Delta x_k(0)} \right|$, where $k = 1, \ldots n$, implies chaos.

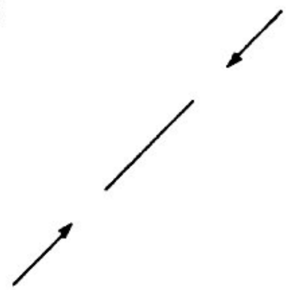
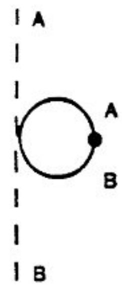
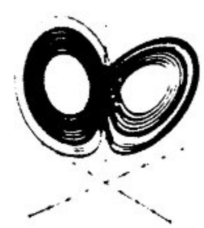
Attractors

An **ATTRACTOR** is a *closed* set A with the properties:

1. A is an INVARIANT SET:
   any trajectory $\mathbf{x}(t)$ that start in A stays in A for ALL time t.
2. A ATTRACTS AN OPEN SET OF INITIAL CONDITIONS:
   there is an open set U containing A ($\subset U$) such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$.
3. A is MINIMAL:
   there is NO proper subset of A that satisfies conditions 1 and 2.

STRANGE ATTRACTOR is an attracting set that is a fractal: has zero measure in the embedding phase space and has FRACTAL dimension. Trajectories within a strange attractor appear to skip around randomly.

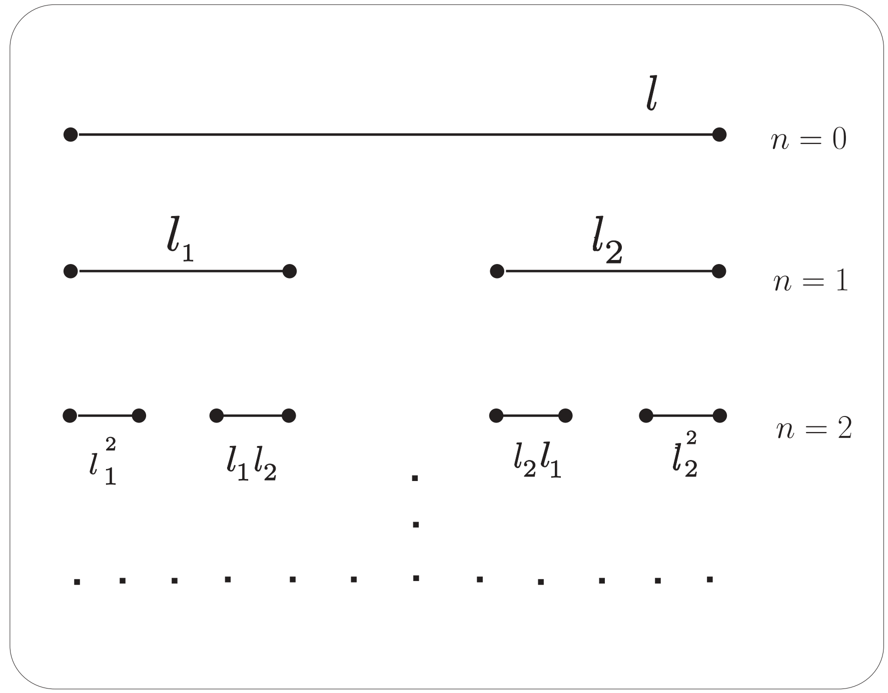
Dynamics on **CHAOTIC ATTRACTOR** exhibits sensitive (exponential) dependence on initial conditions (the 'butterfly' effect).

	ATTRACTOR	LYAPUNOV EXPONENT SPECTRUM
C 	LIMIT CYCLE 	(0,–,–)
D 	POINT 	(–,–,–)
E	STRANGE ATTRACTOR 	(+,0,–)

Fractals and Multifractals

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally *self-similar* and independent of scale (fractal dimension).

A **multifractal** is a set of intertwined fractals. Self-similarity of multifractals is scale dependent (spectrum of dimensions). A deviation from a strict self-similarity is also called **intermittency**.



Two-scale **Cantor** set.

Fractal

A measure (volume) V of a set as a function of size l

$$V(l) \sim l^{D_{\mathrm{F}}}$$

The number of elements of size l needed to cover the set

$$N(l) \sim l^{-D_{\mathrm{F}}}$$

The fractal dimension

$$D_{\mathrm{F}} = \lim_{l \to 0} \frac{\ln N(l)}{\ln 1/l}$$

Multifractal

A (probability) measure versus singularity strength, α

$$p_i(l) \propto l^{\alpha_i}$$

The number of elements in a small range from α to $\alpha + \mathrm{d}\alpha$

$$N_l(\alpha) \sim l^{-f(\alpha)}$$

The multifractal singularity spectrum

$$f(\alpha) = \lim_{\varepsilon \to 0} \lim_{l \to 0} \frac{\ln[N_l(\alpha+\varepsilon) - N_l(\alpha-\varepsilon)]}{\ln 1/l}$$

The generalized dimension

$$D_q = \frac{1}{q-1} \lim_{l \to 0} \frac{\ln \sum_{k=1}^{N} (p_k)^q}{\ln l}$$

Multifractal Characteristics

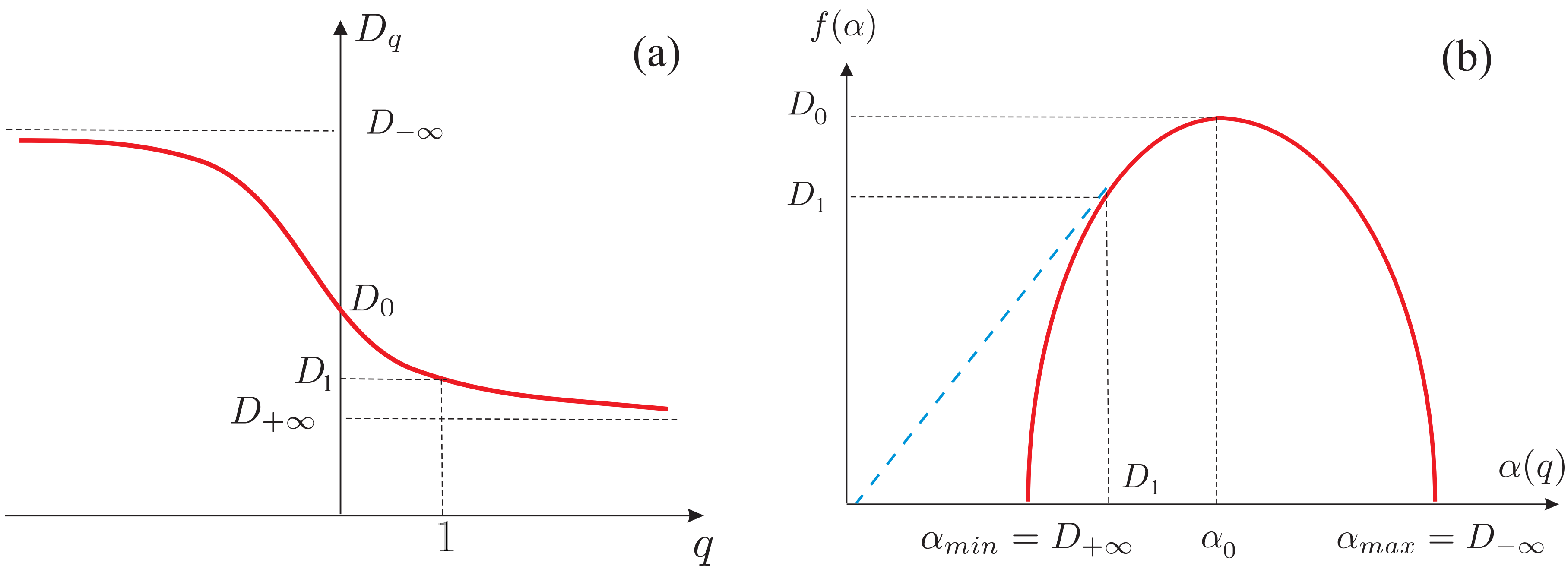


Fig. 1. (a) The generalized dimensions D_q as a function of any real q, $-\infty < q < \infty$, and (b) the singularity multifractal spectrum $f(\alpha)$ versus the singularity strength α with some general properties: (1) the maximum value of $f(\alpha)$ is D_0; (2) $f(D_1) = D_1$; and (3) the line joining the origin to the point on the $f(\alpha)$ curve where $\alpha = D_1$ is tangent to the curve (Ott *et al.*, 1994).

Generalized Scaling Property

The generalized dimensions are important characteristics of *complex* dynamical systems; they quantify multifractality of a given system (Ott, 1993).

Using ($\sum p_i{}^q \equiv \langle p_i{}^{q-1} \rangle_{\rm av}$) a generalized average probability measure of cascading eddies

$$\bar{\mu}(q,l) \equiv \sqrt[q-1]{\langle (p_i)^{q-1} \rangle_{\rm av}} \tag{3}$$

we can identify D_q as scaling of the measure with size l

$$\bar{\mu}(q,l) \propto l^{D_q} \tag{4}$$

Hence, the slopes of the logarithm of $\bar{\mu}(q,l)$ of Eq. (4) versus $\log l$ (normalized) provides

$$D_q = \lim_{l \to 0} \frac{\log \bar{\mu}(q,l)}{\log l} \tag{5}$$

Measures and Multifractality

Similarly, we define a one-parameter q family of (normalized) generalized pseudoprobability measures (Chhabra and Jensen, 1989; Chhabra *et al.*, 1989)

$$\mu_i(q,l) \equiv \frac{p_i^q(l)}{\sum_{i=1}^N p_i^q(l)} \tag{6}$$

Now, with an associated fractal dimension index $f_i(q,l) \equiv \log \mu_i(q,l) / \log l$ for a given q the multifractal spectrum of dimensions is defined directly as the averages taken with respect to the measure $\mu(q,l)$ in Eq. (6) denoted by $\langle \ldots \rangle$

$$f(q) \; \equiv \; \lim_{l \to 0} \sum_{i=1}^N \mu_i(q,l) \; f_i(q,l) = \lim_{l \to 0} \; \frac{\langle \log \mu_i(q,l) \rangle}{\log(l)} \tag{7}$$

and the corresponding average value of the singularity strength is given by (Chhabra and Jensen, 1987)

$$\alpha(q) \; \equiv \; \lim_{l \to 0} \sum_{i=1}^N \mu_i(q,l) \; \alpha_i(l) = \lim_{l \to 0} \; \frac{\langle \log p_i(l) \rangle}{\log(l)}. \tag{8}$$

Turbulence Cascade

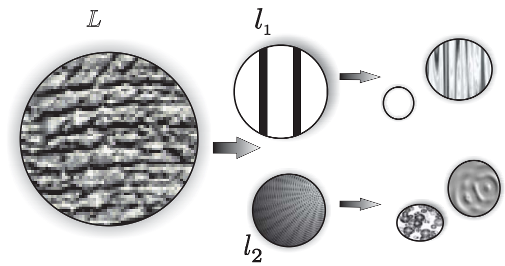


Fig. 1. Schematics of binomial multiplicative processes of cascading eddies. A large eddy of size L is divided into two smaller *not necessarily equal* pieces of size l_1 and l_2. Both pieces may have different probability measures, as indicated by the different shading. At the n-th stage we have 2^n various eddies. The processes continue until the Kolmogorov scale is reached (Meneveau and Sreenivasan, 1991; Macek *et al.*, 2009).

In the case of turbulence cascade these generalized measures are related to inhomogeneity with which the energy is distributed between different eddies (Meneveau and Sreenivasan, 1991). In this way they provide information about dynamics of multiplicative process of cascading eddies. Here high positive values of $q > 1$ emphasize regions of intense fluctuations larger than the average, while negative values of q accentuate fluctuations lower than the average (cf. Burlaga 1995).

Methods of Data Analysis

Structure Functions Scaling

In the inertial range ($\eta \ll l \ll L$) the averaged standard *q*th order ($q > 0$) structure function is scaling with a scaling exponent $\xi(q)$ as

$$S_u^q(l) = \langle |u(x+l) - u(x)|^q \rangle_{\mathrm{av}} \propto l^{\xi(q)} \tag{9}$$

where $u(x)$ and $u(x+l)$ are velocity components parallel to the longitudinal direction separated from a position x by a distance l.

The existence of an inertial range for the experimental data is discussed by Horbury *et al.* (1997), Carbone (1994), and Szczepaniak and Macek (2008).

Energy Transfer Rate and Probability Measures

$$\varepsilon(x,l) \sim \frac{|u(x+l)-u(x)|^3}{l}, \tag{10}$$

To each ith eddy of size l in turbulence cascade ($i = 1,\ldots,N = 2^n$) we associate a probability measure

$$p(x_i,l) \equiv \frac{\varepsilon(x_i,l)}{\sum_{i=1}^{N}\varepsilon(x_i,l)} = p_i(l). \tag{11}$$

This quantity can roughly be interpreted as a probability that the energy is transferred to an eddy of size $l = \upsilon_{\rm sw} t$.

As usual the time-lags can be interpreted as longitudinal separations, $x = \upsilon_{sw} t$ (Taylor's hypothesis).

Magnetic Field Strength Fluctuations and Generalized Measures

Given the normalized time series $B(t_i)$, where $i = 1, \ldots, N = 2^n$ (we take n = 8), to each interval of temporal scale Δt (using $\Delta t = 2^k$, with $k = 0, 1, \ldots, n$) we associate some probability measure

$$p(x_j, l) \equiv \frac{1}{N} \sum_{i=1+(j-1)\Delta t}^{j\Delta t} B(t_i) = p_j(l), \tag{12}$$

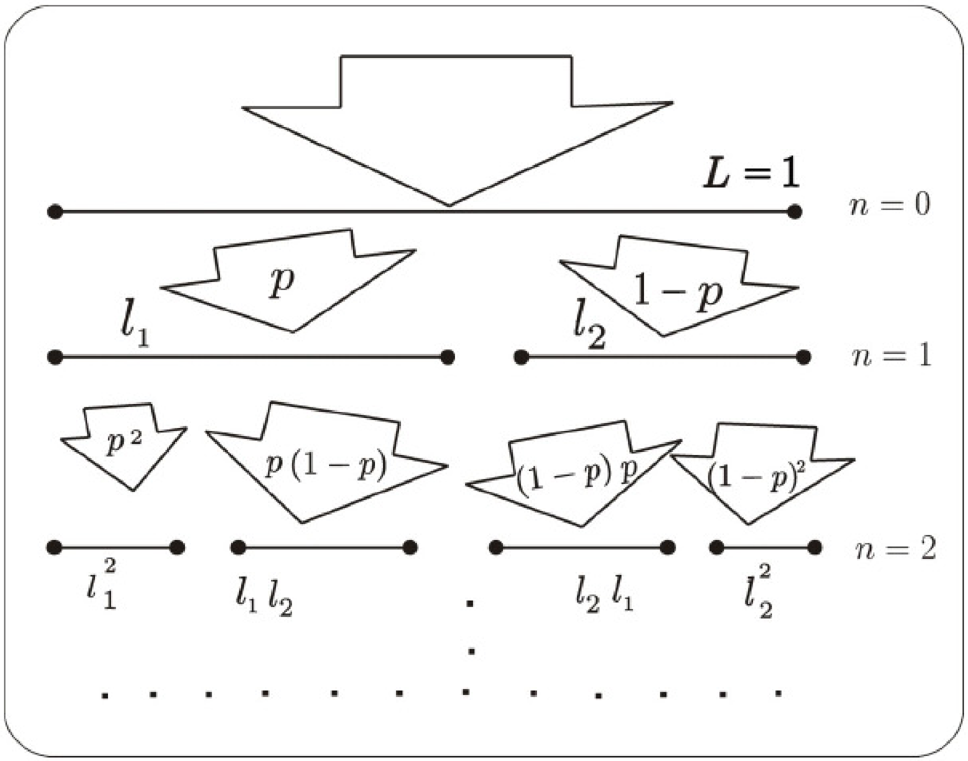
where $j = 2^{n-k}$, i.e., calculated by using the successive (daily) average values $\langle B(t_i, \Delta t) \rangle$ of $B(t_i)$ between t_i and $t_i + \Delta t$. At a position $x = v_{\mathrm{sw}} t$, at time t, where v_{sw} is the average solar wind speed, this quantity can be interpreted as a probability that the magnetic flux is transferred to a segment of a spatial scale $l = v_{\mathrm{sw}} \Delta t$ (Taylor's hypothesis).

The average value of the qth moment of the magnetic field strength B should scale as

$$\langle B^q(l) \rangle \sim l^{\gamma(q)}, \tag{13}$$

with the exponent $\gamma(q) = (q-1)(D_q - 1)$ as shown by Burlaga et al. (1995).

Mutifractal Models for Turbulence



$p_1 + p_2 = 1$

Two-scale model

$l_1 + l_2 \le 1, \;\; l_1 \ne l_2$

One-scale model

$l_1 = l_2 = s \le 1$

P-model

$l_1 = l_2 = \frac{1}{2}$

$$\Gamma^q{}_n(p_1, p_2, l_1, l_2) = \left(\frac{p_1{}^q}{l_1{}^{(q-1)D_q}} + \frac{p_2{}^q}{l_2{}^{(q-1)D_q}} \right)^n = 1$$

Fig. 1. Generalized two-scale Cantor set model for turbulence (Macek, 2007).

Degree of Multifractality and Asymmetry

The difference of the maximum and minimum dimension (the least dense and most dense points in the phase space) is given, e.g., by Macek (2006, 2007)

$$\Delta \equiv \alpha_{\max} - \alpha_{\min} = D_{-\infty} - D_{\infty} = \left| \frac{\log(1-p)}{\log l_2} - \frac{\log(p)}{\log l_1} \right|. \tag{14}$$

In the limit $p \to 0$ this difference rises to infinity (degree of multifractality).

The degree of multifractality Δ is simply related to the deviation from a simple self-similarity. That is why Δ is also a measure of intermittency, which is in contrast to self-similarity (Frisch, 1995, chapter 8).

Using the value of the strength of singularity α_0 at which the singularity spectrum has its maximum $f(\alpha_0) = 1$ we define a measure of asymmetry by

$$A \equiv \frac{\alpha_0 - \alpha_{\min}}{\alpha_{\max} - \alpha_0}. \tag{15}$$

Importance of Multifractality

The concept of multiscale multifractality is of great importance for space plasmas because it allows us to look at intermittent turbulence in the solar wind (e.g., Marsch and Tu, 1997; Bruno *et al.*, 2001). Starting from seminal works of Kolmogorov (1941) and Kraichnan (1965) many authors have attempted to recover the observed scaling exponents, using multifractal phenomenological models of turbulence describing distribution of the energy flux between cascading eddies at various scales (Meneveau and Sreenivasan, 1987, Carbone, 1993, Frisch, 1995).

In particular, the multifractal spectrum has been investigated using Voyager (magnetic field fluctuations) data in the outer heliosphere (e.g., Burlaga, 1991, 1995, 2001) and using Helios (plasma) data in the inner heliosphere (e.g., Marsch *et al.*, 1996; Macek and Szczepaniak, 2008). We have also analysed the multifractal spectrum directly on the solar wind attractor and have shown that it is consistent with that for the multifractal measure of a two-scale weighted Cantor set (Macek, 2007).

The multifractal scaling has also been tested using Ulysses observations (Horbury *et al.*, 1997) and with Advanced Composition Explorer (ACE) and WIND data (e.g., Hnat *et al.*, 2003; Kiyani *et al.*, 2007; Wawrzaszek and Macek, 2010).

Our Multifractal Model of Turbulence

- To quantify scaling of solar wind turbulence, we have developed a generalized weighted two-scale Cantor set model using the partition technique (Szczepaniak and Macek, 2008).

- We have studied the inhomogeneous rate of the transfer of the energy flux indicating multifractal and intermittent behaviour of solar wind turbulence in the inner and outer heliosphere using fluctuations of the velocity of the flow of the solar wind at small scales. We have investigated the resulting spectrum of generalized dimensions and the corresponding multifractal singularity spectrum depending on one probability measure parameter and two rescaling parameters. (Macek and Szcepaniak, 2008; Macek and Wawrzaszek, 2009)

- We have further examined the question of scaling properties of intermittent fluctuations of the magnetic field embedded in the solar wind on both small and large scales using our weighted two-scale Cantor set model in comparison with the simple one scale multifractal spectrum (Szczepaniak and Macek, 2008).

- In particular, we have shown that the degree of multifractality for fluctuations of the interplanetary magnetic field strengths at small scales is smaller than that at the large scales. Moreover, we demonstrate that the multifractal spectrum is asymmetric for small scale fluctuations, in contrast to the rather symmetric spectrum observed on large scales. It is worth noting that for the multifractal two-scale Cantor set model a good agreement with the data is obtained. Hence we propose this new model as a useful tool for analysis of intermittent fluctuations of the interplanetary magnetic field strength on both small and large scales (Macek and Wawrzaszek, 2010).

Observation of the Multifractal Spectrum at the Frontiers of the Solar System

- In addition, the times series of the magnetic field strengths measured in situ by Voyager 1 spacecraft to very large distances from the Sun and even in the heliosheath have already been analysed. It is known that fluctuations of the solar magnetic fields at large scales from 2 to 16 (32) days may exhibit multifractal scaling laws (Burlaga, 2004).

- In 2004 at distances of 85 AU from the Sun Voyager 1 crossed the termination heliospheric shock separating the Solar System plasma from the surrounding heliosheath, and entered the subsonic solar wind, where it has encountered quite unusual conditions The magnetic fields in the heliosheath are normally distributed in contrast to the lognormal distribution in the outer heliosphere (Burlaga, 2005). It also appears that the magnetic field in the inner heliosheath, at $\sim$95 AU, has a multifractal structure on large scales from $\sim$2 to 16 days (Burlaga *et al.*, 2006).

- We also have examined the question of scaling properties of intermittent solar wind turbulence on large scales using our weighted two-scale Cantor set model. We compare the Burlaga's (2004) fit to the multifractal spectrum with the two-scale model at the heliospheric boundary (Macek and Wawrzaszek, 2009).

- We have shown that the degree of multifractality for fluctuations of the interplanetary magnetic field strength before shock crossing is greater than that in the heliosheath. Moreover, we have demonstrated that the multifractal spectrum is asymmetric before shock crossing, in contrast to the symmetric spectrum observed in the heliosheath; the solar wind in the outer heliosphere may exhibit strong asymmetric scaling (Macek and Wawrzaszek, 2010a, Macek et al., 2011).

Conclusions

- Fratal structure can describe complex shapes in the real word.
- Nonlinear systems exhibit complex phenomena, including chaos.
- Strange chaotic attractors has fractal structure and are sensitive to initial conditions.
- Within the complex dynamics of the fluctuating intermittent parameters of turbulent media there is a detectable, hidden order described by a generalized Cantor set that exhibits a multifractal structure.

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