Nonlinear and Fractals Analysis

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Objective

The aim of the course is to give students an introduction to the new developments in nonlinear dynamics and fractals. Emphasis will be on the basic concepts of stability, bifurcations and intermittency, based on intuition rather than mathematical proofs. The specific exercises will also include applications to physics, astrophysics and space physics, chemistry, biology, and even economy. On successful completion of this course, students should understand and apply the theory to simple nonlinear dynamical systems and be able to evaluate the importance of nonlinearity in various environments.

Plan of the Course

1. Introduction
   - Dynamical and Geometrical View of the World
   - Fractals
   - Stability of Linear Systems

2. Nonlinear Dynamics
   - Attracting and Stable Fixed Points
   - Nonlinear Systems: Pendulum

3. Fractals and Chaos
   - Strange Attractors and Deterministic Chaos
   - Bifurcations

4. Strange Attractors
    - Stretching and Folding Mechanism
    - Baker's Map
    - Logistic Map
    - Hénon Map

5. Multifractals
    - Intermittent Turbulence
    - Weighted Two-Scale Cantors Set
    - Multifractals Analysis of Turbulence

6. Conclusion: importance of nonlinearity and fractals

Dynamics - A Capsule History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff Kolmogorov Arnol'd Moser	Complex behavior in Hamiltonian mechanics
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

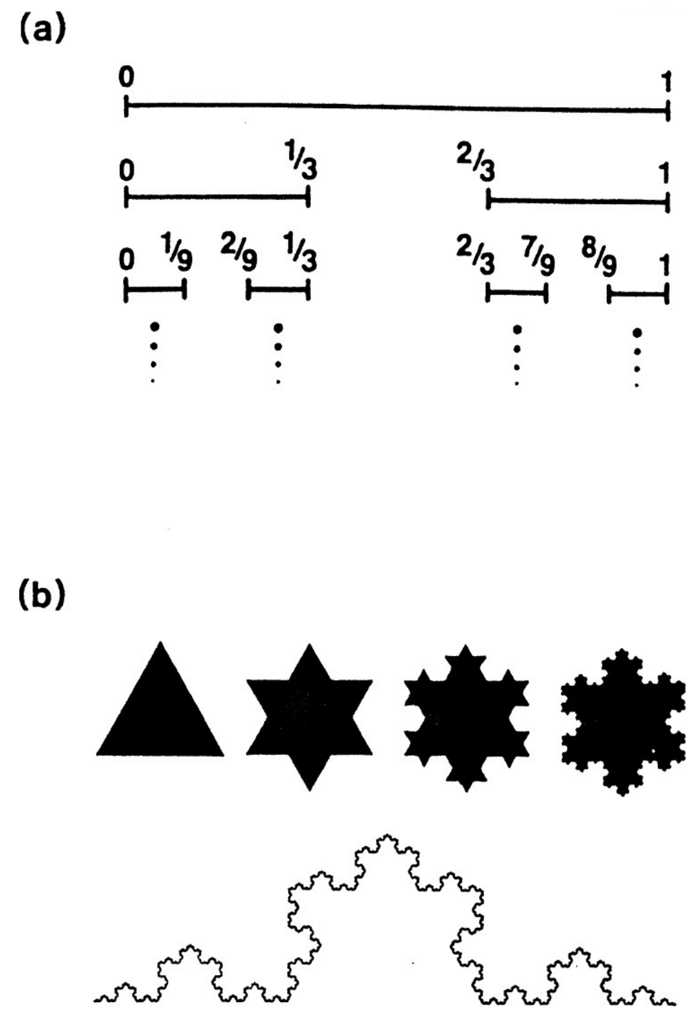
Number of variables →

Nonlinearity ↓	$n = 1$	$n = 2$	$n \geq 3$	$n >> 1$	Continuum
Linear	*Growth, decay, or equilibrium*	*Oscillations*		*Collective phenomena*	*Waves and patterns*
	Exponential growth	Linear oscillator	Civil engineering, structures	Coupled harmonic oscillators	Elasticity
	RC circuit	Mass and spring	Electrical engineering	Solid-state physics	Wave equations
	Radioactive decay	RLC circuit		Molecular dynamics	Electromagnetism (Maxwell)
		2-body problem (Kepler, Newton)		Equilibrium statistical mechanics	Quantum mechanics (Schrödinger, Heisenberg, Dirac)
					Heat and diffusion
					Acoustics
					Viscous fluids
			The frontier		
Nonlinear			*Chaos*		*Spatio-temporal complexity*
	Fixed points	Pendulum	Strange attractors (Lorenz)	Coupled nonlinear oscillators	Nonlinear waves (shocks, solitons)
	Bifurcations	Anharmonic oscillators	3-body problem (Poincaré)	Lasers, nonlinear optics	Plasmas
	Overdamped systems, relaxational dynamics	Limit cycles	Chemical kinetics	Nonequilibrium statistical mechanics	Earthquakes
	Logistic equation for single species	Biological oscillators (neurons, heart cells)	Iterated maps (Feigenbaum)	Nonlinear solid-state physics (semiconductors)	General relativity (Einstein)
		Predator-prey cycles	Fractals (Mandelbrot)	Josephson arrays	Quantum field theory
		Nonlinear electronics (van der Pol, Josephson)	Forced nonlinear oscillators (Levinson, Smale)	Heart cell synchronization	Reaction-diffusion, biological and chemical waves
			Practical uses of chaos	Neural networks	Fibrillation
			Quantum chaos ?	Immune system	Epilepsy
				Ecosystems	Turbulent fluids (Navier-Stokes)
				Economics	Life

Fractals

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole.

Fractals are generally *self-similar* and independent of scale (fractal dimension).



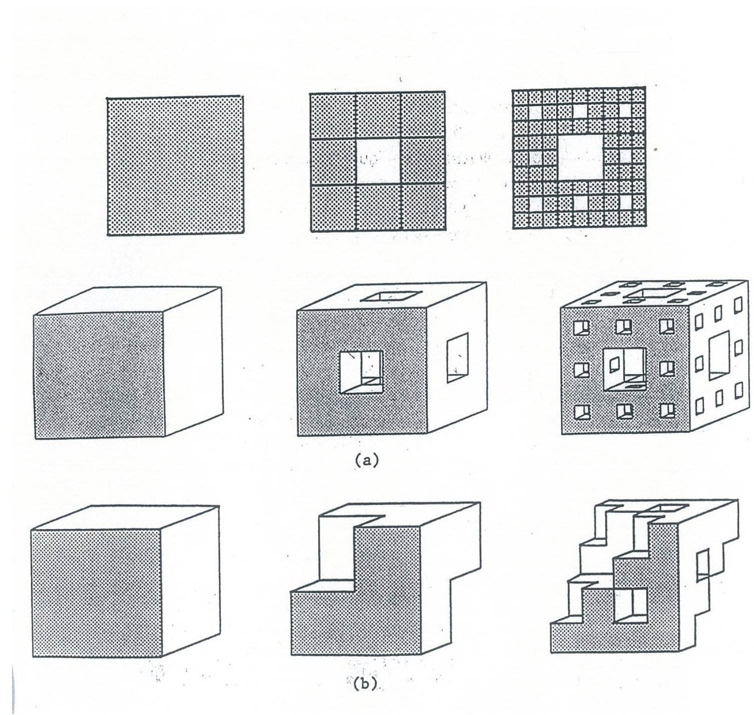
If N_n is the number of elements of size r_n needed to cover a set (C is a constant) is:

$$N_n = \frac{C}{{r_n}^D}, \tag{1}$$

then in case of self-similar sets:
$N_{n+1} = C/(r_{n+1})^D$,
and hence the fractal similarity dimension D is

$$D = \ln(N_{n+1}/N_n)/\ln(r_n/r_{n+1}). \tag{2}$$

- Cantor set $D = \ln 2/\ln 3$
- Koch curve $D = \ln 4/\ln 3$
- Sierpinski carpet $D = \ln 8/\ln 3$
- Mengor sponge $D = \ln 20/\ln 3$
- Fractal cube $D = \ln 6/\ln 2$



Stability of Linear Systems

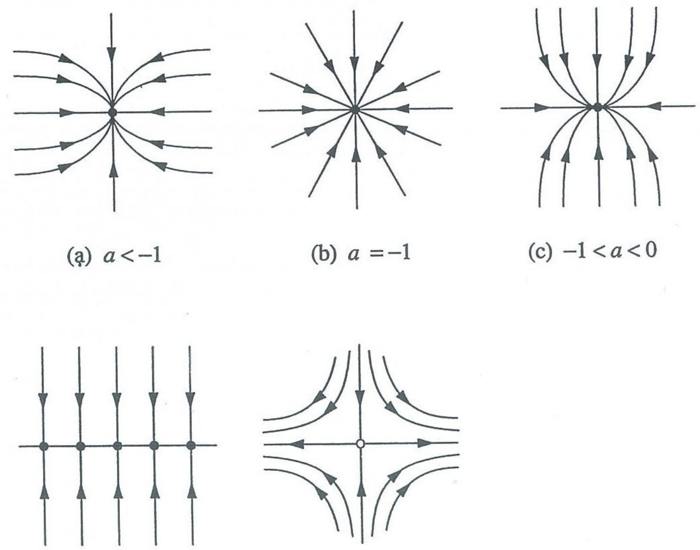
Two-Dimensional System

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solutions

$$x(t) = x_o \mathrm{e}^{at}$$

$$y(t) = y_o \mathrm{e}^{-t}$$



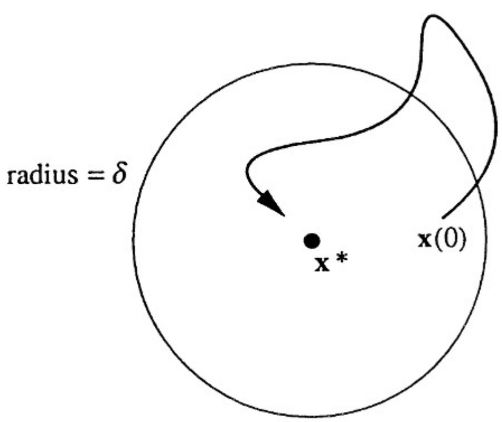
(a) $a < -1$ (b) $a = -1$ (c) $-1 < a < 0$ (d) $a = 0$ (e) $a > 0$

Attracting and Stable Fixed Points

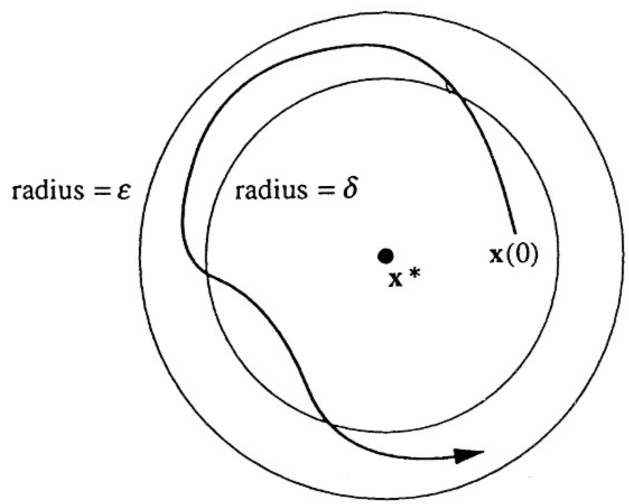
We consider a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, where $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$.

We say that $\mathbf{x}^*$ is *attracting* if there is a $\delta > 0$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$ whenever $|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$: any trajectory that starts within a distance δ of $\mathbf{x}^*$ is guaranteed to converge to $\mathbf{x}^*$.

A fixed point $\mathbf{x}^*$ is *Lyapunov stable* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$ whenever $t \geq 0$ and $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$: all trajectories that start within δ of $\mathbf{x}^*$ remain within ε of $\mathbf{x}^*$ for all positive time.

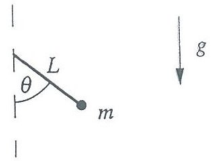


Attracting



Liapunov stable

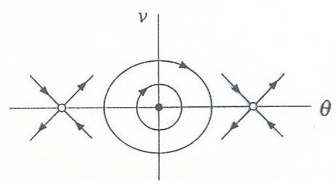
Nonlinear Systems: Pendulum

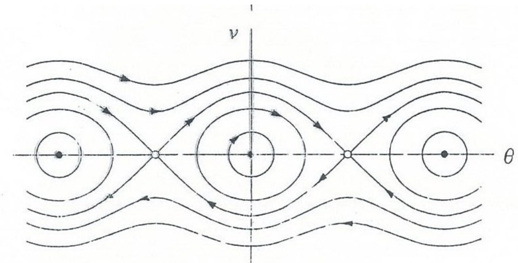


$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \qquad \ddot{\theta} + \sin\theta = 0 \qquad \begin{aligned}\dot{\theta} &= v \\ \dot{v} &= -\sin\theta\end{aligned}$$

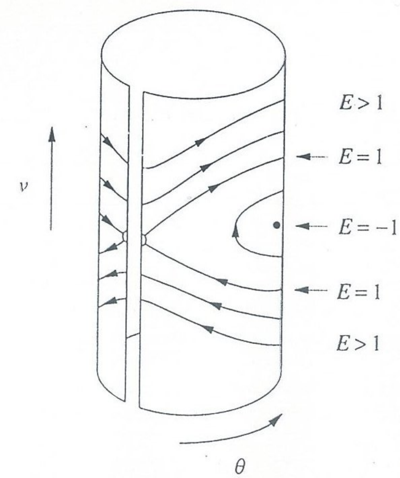
The energy function

$$E(\theta, v) = \tfrac{1}{2}v^2 - \cos\theta$$





Cylindrical Phase Space



Attractors

An **ATTRACTOR** is a *closed* set A with the properties:

1. A is an INVARIANT SET:
   any trajectory $\mathbf{x}(t)$ that start in A stays in A for ALL time t.
2. A ATTRACTS AN OPEN SET OF INITIAL CONDITIONS:
   there is an open set U containing A ($\subset U$) such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$.
3. A is MINIMAL:
   there is NO proper subset of A that satisfies conditions 1 and 2.

STRANGE ATTRACTOR is an attracting set that is a fractal: has zero measure in the embedding phase space and has FRACTAL dimension. Trajectories within a strange attractor appear to skip around randomly.

Dynamics on **CHAOTIC ATTRACTOR** exhibits sensitive (exponential) dependence on initial conditions (the 'butterfly' effect).

Deterministic Chaos

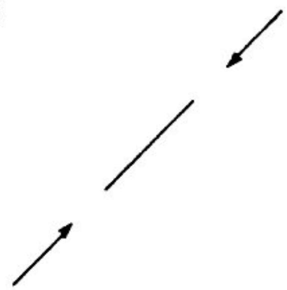
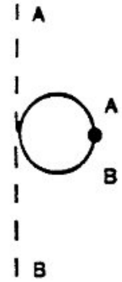
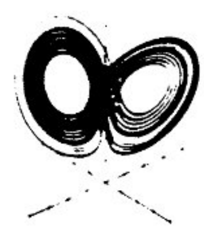
CHAOS ($\chi\alpha o\varsigma$) is

- NON-PERIODIC long-term behavior
- in a DETERMINISTIC system
- that exhibits SENSITIVITY TO INITIAL CONDITIONS.

We say that a bounded solution $\mathbf{x}(t)$ of a given dynamical system is SENSITIVE TO INITIAL CONDITIONS if there is a finite fixed distance $r > 0$ such that for any neighborhood $\|\Delta\mathbf{x}(0)\| < \delta$, where $\delta > 0$, there exists (at least one) other solution $\mathbf{x}(t) + \Delta\mathbf{x}(t)$ for which for some time $t \geq 0$ we have $\|\Delta\mathbf{x}(t)\| \geq r$.

There is a fixed distance r such that no matter how precisely one specify an initial state there is a nearby state (at least one) that gets a distance r away.

Given $\mathbf{x}(\mathbf{t}) = \{x_1(t), \ldots, x_n(t)\}$ any positive finite value of Lyapunov exponents $\lambda_k = \lim_{t\to\infty} \frac{1}{t} \ln \left| \frac{\Delta x_k(t)}{\Delta x_k(0)} \right|$, where $k = 1, \ldots n$, implies chaos.

	ATTRACTOR	LYAPUNOV EXPONENT SPECTRUM
C 	LIMIT CYCLE 	(0,−,−)
D 	POINT 	(−,−,−)
E	STRANGE ATTRACTOR 	(+,0,−)

Types of Bifurcations

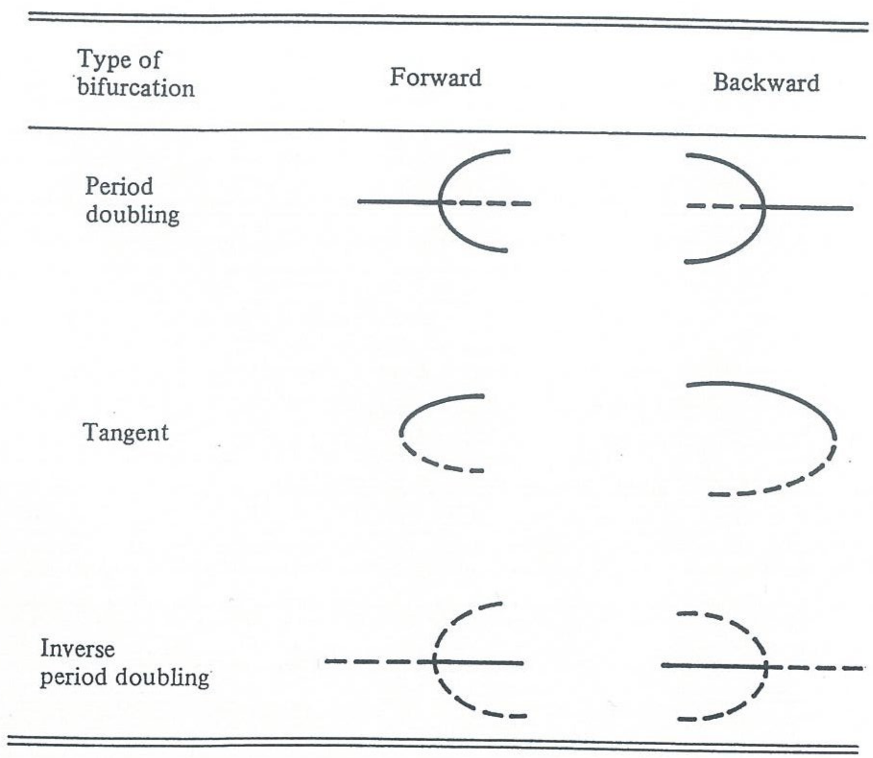
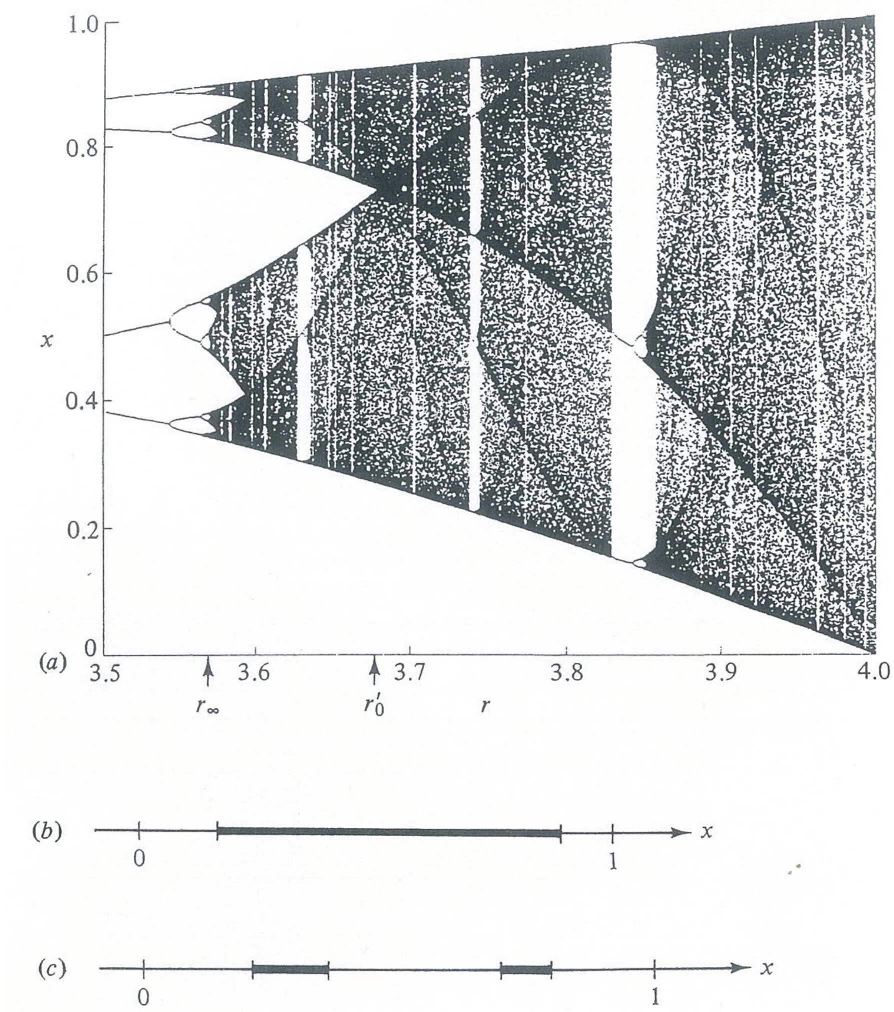


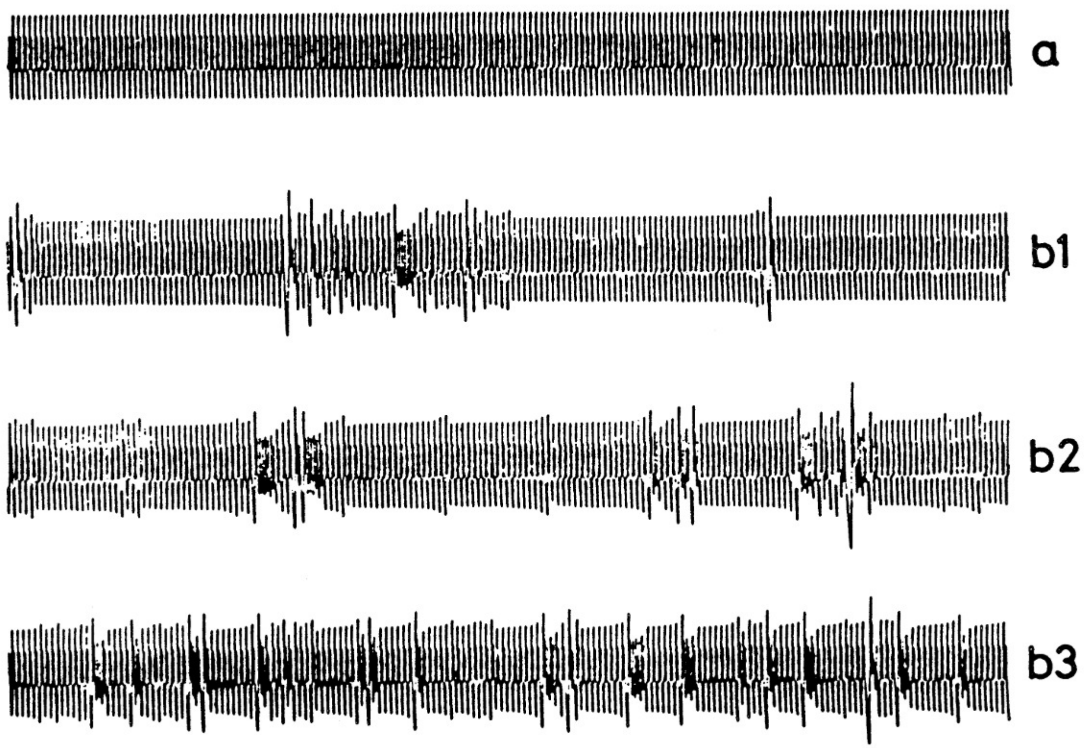
Figure 2.15 Generic bifurcations of differentiable one-dimensional maps. The system parameter r increases toward the right. The vertical scale represents the value of the map variable. Dashed lines represent unstable orbits. Solid lines represent stable orbits.

Bifurcation Diagram for the Logistic Map



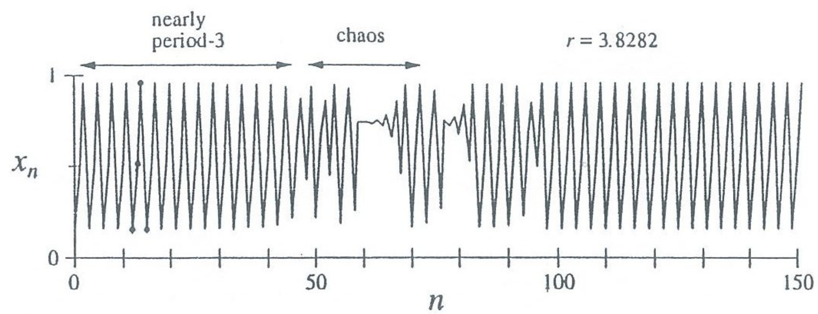
Intermittency

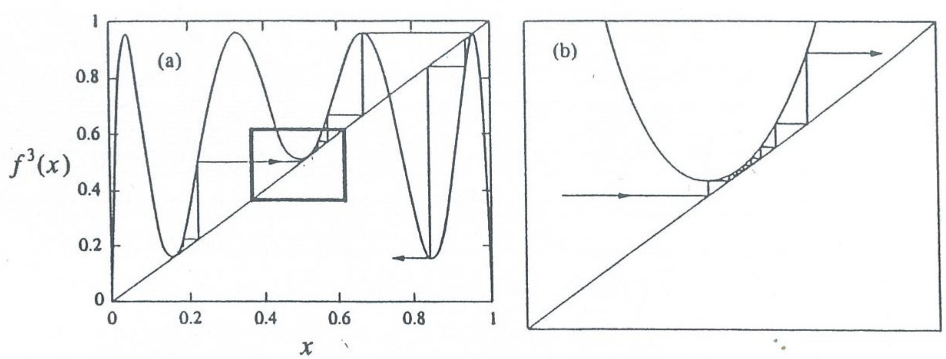
In dynamical systems theory: occurrence of a signal that alternates randomly between long periods of regular behavior and relatively short irregular bursts. In other words, motion in intermittent dynamical system is nearly periodic with occasional irregular bursts.



Pomeau & Manneville, 1980

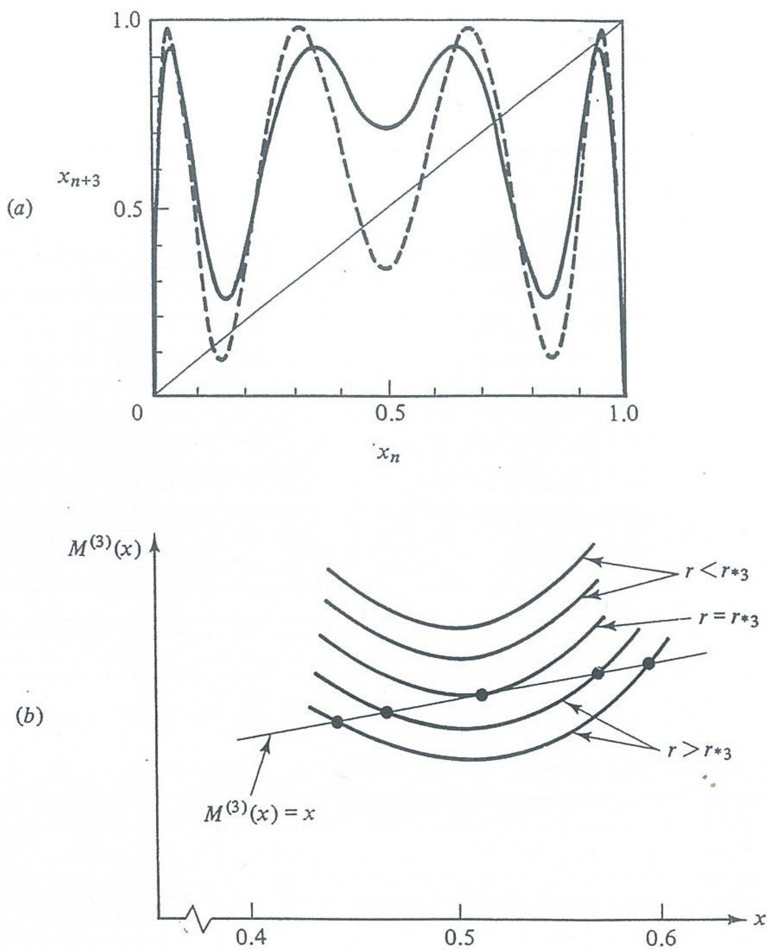
Intermittent Behavior

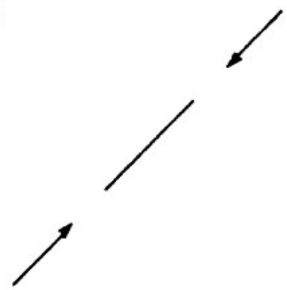
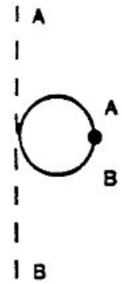




Bifurcation and Intermittency

Figure 2.13 (a) x_{n+3} versus x_n from $r = 3.7 < r_{*3}$ (solid curve) and for $r = 3.9 > r_{*3}$ (dashed curve). (b) Schematic of $M^3(x)$ versus x near $x = 0.5$.



	ATTRACTOR	LYAPUNOV EXPONENT SPECTRUM
C 	LIMIT CYCLE 	(0,−,−)
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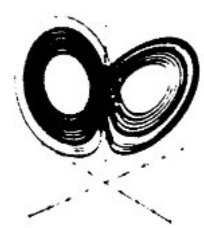
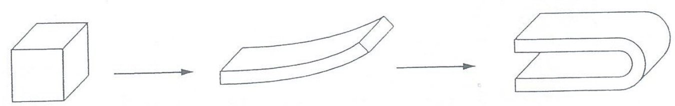
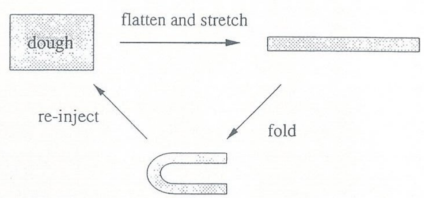
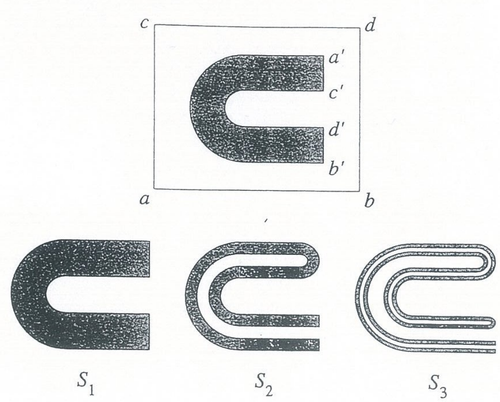
E	STRANGE ATTRACTOR 	(+,0,−)

Horseshoe Map





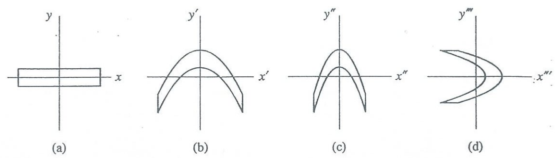


Henon Map

The ***Hénon map*** is given by

$$x_{n+1} = y_n + 1 - ax_n^2 \,, \quad y_{n+1} = bx_n \,,$$

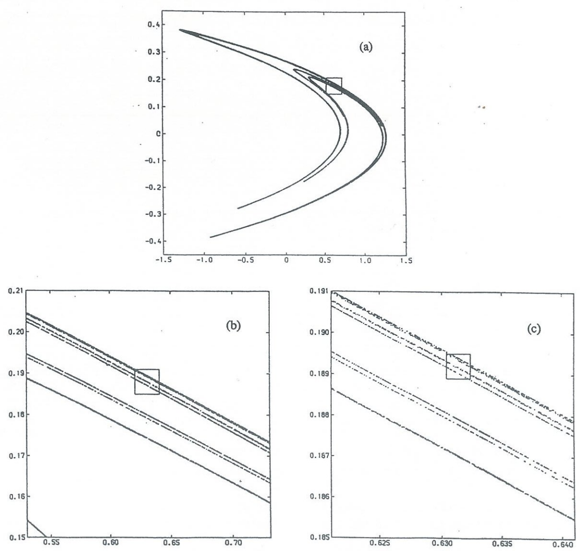
where a and b are adjustable parameters.



$T' : \; x' = x, \quad y' = 1 + y - ax^2.$

$T'' : \; x'' = bx', \quad y'' = y'$

$T''' : \; x''' = y'', \quad y''' = x''.$

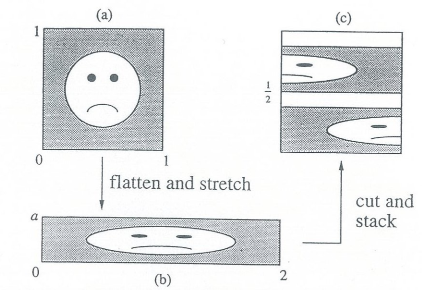


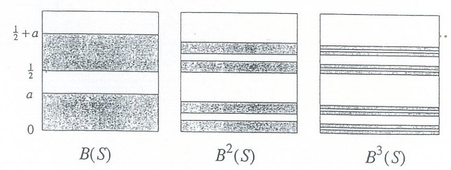
Baker's Map

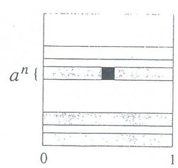
The *baker's map* B of the square $0 \le x \le 1$, $0 \le y \le 1$ to itself is given by

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n,\ ay_n) & \text{for } 0 \le x_n < \frac{1}{2} \\ (2x_n - 1,\ ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases}$$

where a is a parameter in the range $0 < a \le \frac{1}{2}$.





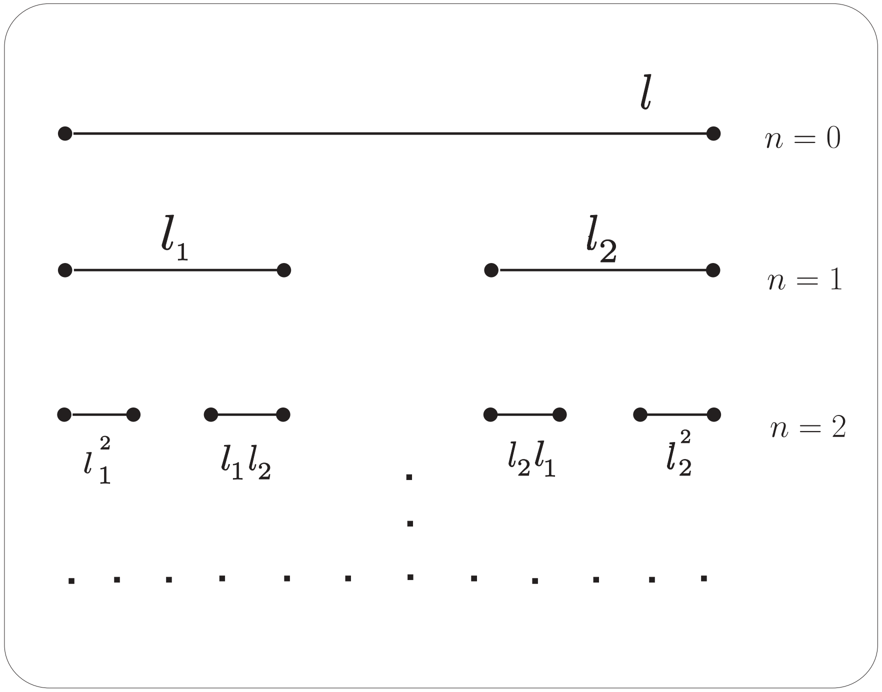


$$d = \lim_{\varepsilon \to 0} \frac{\ln N}{\ln(\frac{1}{\varepsilon})} = \lim_{n \to \infty} \frac{\ln[(a/2)^{-n}]}{\ln(a^{-n})} = 1 + \frac{\ln \frac{1}{2}}{\ln a}.$$

Fractals and Multifractals

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally *self-similar* and independent of scale (fractal dimension).

A **multifractal** is a set of intertwined fractals. Self-similarity of multifractals is scale dependent (spectrum of dimensions). A deviation from a strict self-similarity is also called **intermittency**.



Two-scale **Cantor** set.

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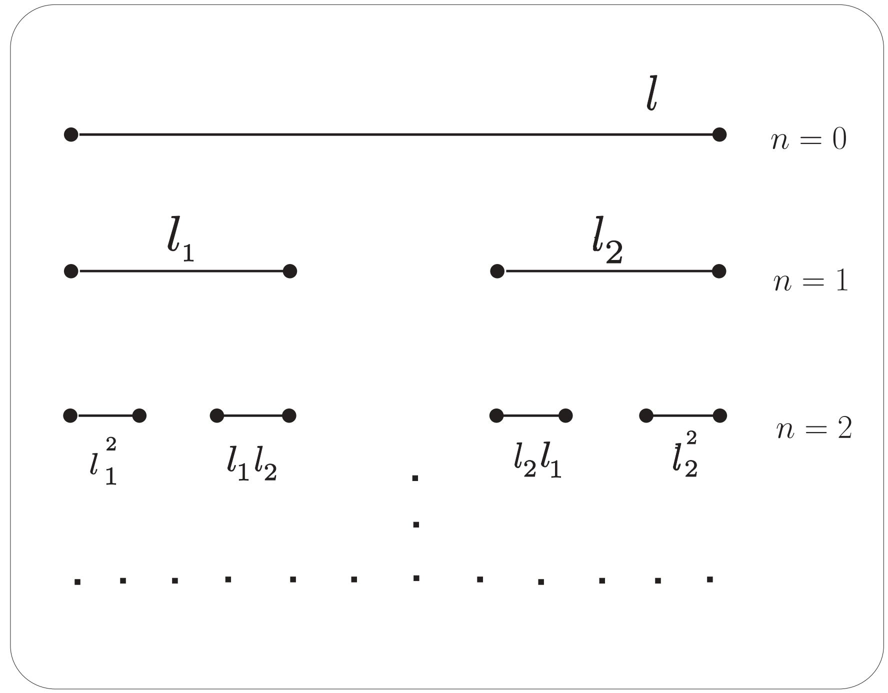


Two-scale **Cantor** set.

Multifractal Characteristics

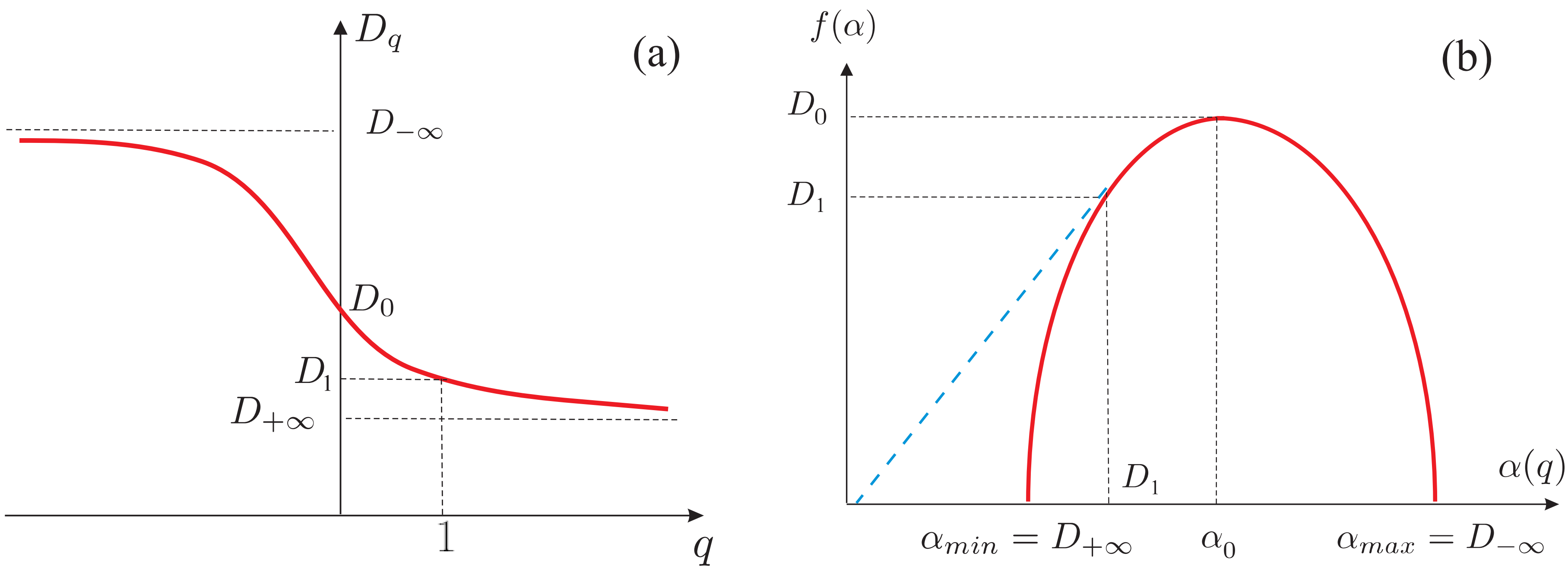


Fig. 1. (a) The generalized dimensions D_q as a function of any real q, $-\infty < q < \infty$, and (b) the singularity multifractal spectrum $f(\alpha)$ versus the singularity strength α with some general properties: (1) the maximum value of $f(\alpha)$ is D_0; (2) $f(D_1) = D_1$; and (3) the line joining the origin to the point on the $f(\alpha)$ curve where $\alpha = D_1$ is tangent to the curve (Ott *et al.*, 1994).

Turbulence Cascade

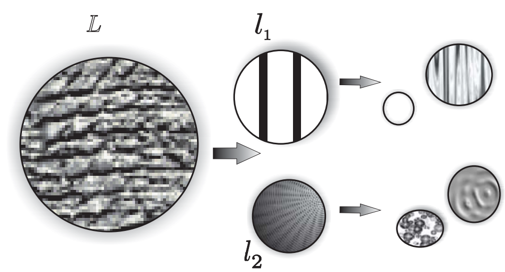
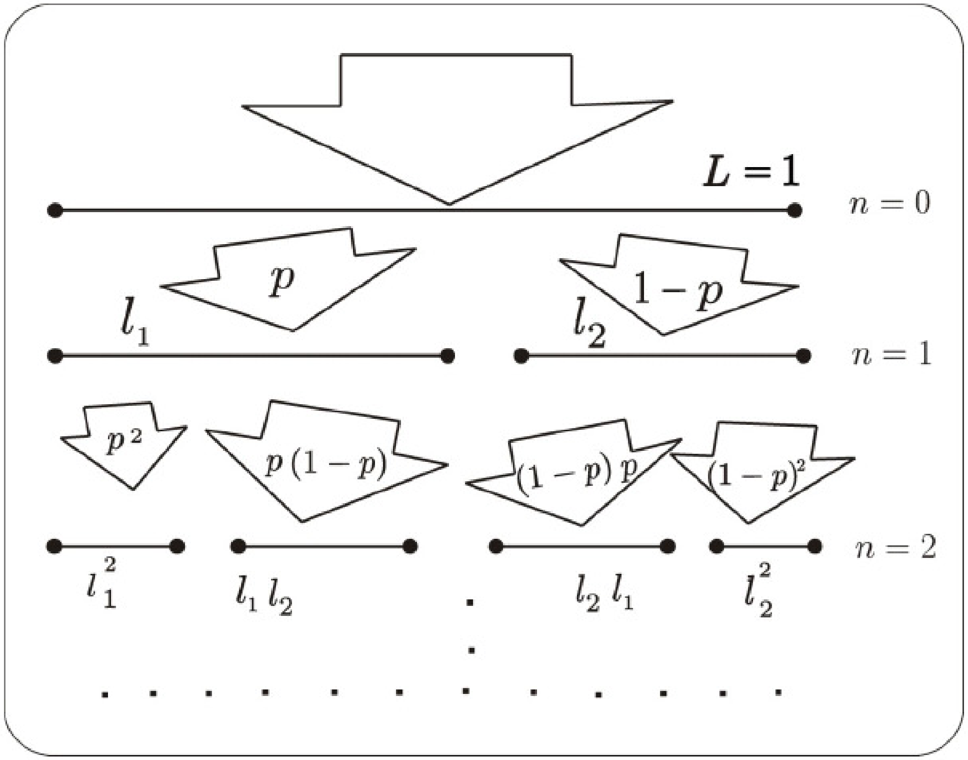
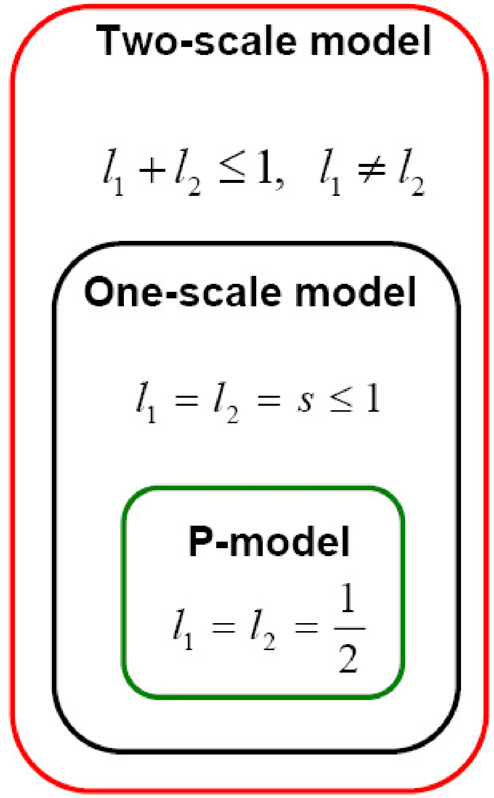


Fig. 1. Schematics of binomial multiplicative processes of cascading eddies. A large eddy of size L is divided into two smaller *not necessarily equal* pieces of size l_1 and l_2. Both pieces may have different probability measures, as indicated by the different shading. At the n-th stage we have 2^n various eddies. The processes continue until the Kolmogorov scale is reached (Meneveau and Sreenivasan, 1991; Macek *et al.*, 2009).

Mutifractal Models for Turbulence



$$p_1 + p_2 = 1$$



$$\Gamma^q{}_n(p_1, p_2, l_1, l_2) = \left(\frac{p_1{}^q}{l_1{}^{(q-1)D_q}} + \frac{p_2{}^q}{l_2{}^{(q-1)D_q}} \right)^n = 1$$

Fig. 1. Generalized two-scale Cantor set model for turbulence (Macek, 2007).

Conclusions

- Fratal structure can describe complex shapes in the real word.
- Nonlinear systems exhibit complex phenomena, including bifurcation, intermittency, and chaos.
- Strange chaotic attractors has fractal structure and are sensitive to initial conditions.
- Within the complex dynamics of the fluctuating intermittent parameters of turbulent media there is a detectable, hidden order described by a generalized Cantor set that exhibits a multifractal structure.

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