JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 114, A03108, doi:10.1029/2008JA013795, 2009

Click Here for Full Article

Evolution of asymmetric multifractal scaling of solar wind turbulence in the outer heliosphere

Wiesław M. Macek[1,2] and Anna Wawrzaszek[2]

Received 3 October 2008; revised 4 December 2008; accepted 19 December 2008; published 21 March 2009.

[1] The aim of this study is to examine the question of scaling properties of intermittent turbulence in the space environment. We analyze time series of velocities of the slow and fast speed streams of the solar wind measured in situ by Voyager 2 spacecraft in the outer heliosphere during solar minimum at various distances from the Sun (2.5, 25, and 50 AU). To quantify asymmetric scaling of solar wind turbulence, we consider a generalized two-scale weighted Cantor set with two different scales describing nonuniform distribution of the kinetic energy flux between cascading eddies of various sizes. We investigate the resulting spectrum of generalized dimensions and the corresponding multifractal singularity spectrum depending on one probability measure parameter and two rescaling parameters, demonstrating that the multifractal scaling is often rather asymmetric. In particular, we show that the degree of multifractality for the solar wind during solar minimum is greater for fast streams' velocity fluctuations than that for the slow streams; the fast wind during solar minimum may exhibit strong asymmetric scaling. Moreover, we observe the evolution of multifractal scaling of the solar wind in the outer heliosphere. It is worth noting that for the model with two different scaling parameters a much better agreement with the solar wind data is obtained, especially for the negative index of the generalized dimensions. Therefore, we argue that there is a need to use a two-scale cascade model. Hence, we propose this model as a useful tool for analysis of intermittent turbulence in various environments, and we hope that our new more general asymmetric multifractal model can shed light on the nature of turbulence.

Citation: Macek, W. M., and A. Wawrzaszek (2009), Evolution of asymmetric multifractal scaling of solar wind turbulence in the outer heliosphere, *J. Geophys. Res.*, *114*, A03108, doi:10.1029/2008JA013795.

1. Introduction

[2] The solar wind provides a unique laboratory for study turbulence in various environments, including space and astrophysical plasmas, see for a review [e.g., *Bruno and Carbone*, 2005]. The slow solar wind most likely originates from equatorial regions of the solar corona. The fast wind is associated with coronal holes and is relatively uniform and stable, while the slow wind is more turbulent and consequently quite variable in terms of velocities. Notwithstanding of progress in MHD and Hall MHD turbulence models the nature of the fluctuations in the solar wind plasma parameters is still not sufficiently understood. Fortunately, it appears that a certain kind of order does lie concealed within the irregular solar wind fluctuations, which can be described using methods based on fractal analysis [*Burlaga*, 1991; *Carbone*, 1993; *Frisch*, 1995; *Marsch et al.*, 1996; *Macek*, 2006b]. Admittedly, fractal theory does not provide direct inside into the equations describing the dynamics. However, owing to its mathematical rigor and elegance it could provide a natural mathematical language for understanding scaling properties of turbulence [*Meneveau and Sreenivasan*, 1987, 1991; *Frisch*, 1995]. This involves the notions of fractal and multifractal sets, which could help us to look at scaling properties of complex dynamics in a certain state space of a given nonlinear system [*Mandelbrot*, 1989; *Falconer*, 1990; *Ott*, 1993].

[3] We remind that a fractal is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Strange attractors are often fractal sets, which exhibits a hidden order within chaos. Fractals are generally self-similar and independent of scale (generally with a particular fractal dimension). A multifractal is an object that demonstrate various self-similarities, described by a multifractal spectrum of dimensions and a singularity spectrum. One can say that self-similarity of multifractals is point dependent resulting in the singularity spectrum. A multifractal is therefore in a certain sense like a set of intertwined fractals.

[4] The question of multifractality is of great importance for space plasmas because it allows us to look at intermittent turbulence in the solar wind [e.g., *Burlaga*, 1991; *Carbone*,

[1]Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University in Warsaw, Warsaw, Poland.
[2]Space Research Centre, Polish Academy of Sciences, Warsaw, Poland.

0148-0227/09/2008JA013795$09.00

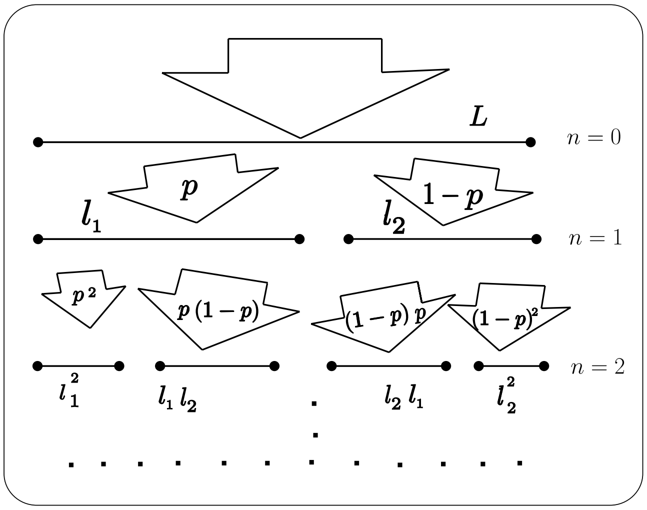

Figure 1. Two-scale weighted Cantor set model for asymmetric solar wind turbulence [*Macek*, 2007].

1993; *Carbone and Bruno*, 1996; *Marsch et al.*, 1996; *Marsch and Tu*, 1997; *Bruno et al.*, 2001]. Starting from *Richardson*'s [1922] scenario of turbulence, many authors still attempt to recover the observed scaling exponents, using some simple and more advanced fractal and multifractal models of turbulence describing distribution of the energy flux between cascading eddies at various scales. In particular, the multifractal spectrum has been investigated using Voyager (magnetic field fluctuations) data in the outer heliosphere [e.g., *Burlaga*, 1991; *Burlaga et al.*, 1993; *Burlaga*, 2001] and using Helios (plasma) data in the inner heliosphere [e.g., *Marsch et al.*, 1996]. The multifractal scaling has also been investigated using Ulysses observations [e.g., *Horbury et al.*, 1997; *Horbury and Balogh*, 2001] and with Advanced Composition Explorer (ACE) and WIND data [e.g., *Hnat et al.*, 2003, 2007; *Kiyani et al.*, 2007; *Szczepaniak and Macek*, 2008].

[5] In general, the spectrum of generalized dimensions D_q as a function of a continuous index, $-\infty < q < \infty$, with a degree of multifractality $\Delta = D_{-\infty} - D_{\infty}$, quantify multifractality of a given system [e.g., *Ott*, 1993]. The degree of multifractality is simply related to the deviation from a simple self-similarity. That is why Δ is also a measure of intermittency, which is in contrast to self-similarity [*Frisch*, 1995, chap. 8]. The related multifractal singularity spectrum $f(\alpha)$ as a function of a singularity strength α is also often used.

[6] The presence of a chaotic strange attractor in the solar wind data has also been discussed in the literature [e.g., *Macek*, 1998; *Macek and Redaelli*, 2000]. On the other hand, it is rather clear that turbulence cannot be reduced to chaotic phenomena. We have also considered the D_q spectrum for the solar wind attractor using a simple multifractal model with a measure of the self-similar weighted Cantor set with one parameter describing uniform compression in phase space and another parameter for the probability measure of the attractor of the system. The spectrum has been found to be consistent with the data, at least for positive index q of the generalized dimensions D_q [*Macek*, 2002, 2003, 2006a; *Macek et al.*, 2005, 2006]. However, the full singularity spectrum is necessary to quantify the degree of multifractality. Notwithstanding of the well-known statistical problems with negative q [*Macek*, 2006a], we have succeeded in estimating the entire experimental spectrum for solar wind attractor using a generalized weighted Cantor set with two different scales describing nonuniform compression [*Macek*, 2007].

[7] Recently, to further look into the multifractal scaling properties of turbulence, we have considered this generalized weighted Cantor set also in the context of turbulence cascade [*Macek and Szczepaniak*, 2008]. In this way we have argued that there is, in fact, need to use a two-scale cascade model. Therefore, we have already investigated the multifractal spectrum depending on two rescaling parameters and one probability measure parameter using Helios 2 data, and in particular, we have demonstrated that intermittent pulses are stronger for asymmetric scaling and a much better agreement with the data is obtained, especially for $q < 0$.

[8] In this paper, we would like to test the degree of multifractality and asymmetry of the multifractal scaling for the wealth of data provided by another space mission. Namely, we further consider in fuller detail the question of scaling properties of intermittent turbulence using velocities of the slow and fast speed streams of the solar wind measured in situ by Voyager 2 at various distances from the Sun (2.5, 25, and 50 AU). By using our cascade model with two different scaling parameters we show that the degree of multifractality of the solar wind in the outer heliosphere during solar minimum is somewhat different for fast solar wind velocity fluctuations than that for the slow solar wind. On the other hand, both the degree of multifractality and the degree of asymmetry are correlated with the heliospheric distance and we observe the evolution of asymmetric multifractal scaling in the outer heliosphere [cf. *Burlaga*, 1991; *Burlaga et al.*, 2003; *Burlaga*, 2004]. Thus, we still hope that this generalized new asymmetric multifractal model could shed light on the nature of turbulence and will be a useful tool for analysis of intermittent turbulence in various environments. This paper is organized as follows. In section 2 a generalized two-scale Cantor set model is introduced, and the data are presented in section 3. The methods related the concept of the generalized dimensions and the singularity spectrum in the context of turbulence scaling are reviewed in section 4. The results of our analysis are presented and discussed in section 5. The importance of our new more general asymmetric multifractal cascade model is underlined in section 6.

2. Theoretical Cascade Model: Two-Scale Cantor Set

[9] An interesting example of multifractals is the Cantor set with weight p and two scales, $l_1 + l_2 \leq L$, schematically shown in Figure 1. Even though one can find this complex geometrical object in many classical textbooks [e.g., *Falconer*, 1990; *Ott*, 1993], it is still difficult to understand complexity of this set that exhibits multifractality in various real systems.

[10] At each stage of construction of this generalized Cantor set we basically have two rescaling parameters l_1 and l_2, where $l_1 + l_2 \leq L = 1$ (normalized) and two different

probability measure $p_1 = p$ and $p_2 = 1 - p$. To obtain the generalized dimensions $D_q \equiv \tau(q)/(q - 1)$ for this multifractal set we use the following partition function (a generator) at the nth level of construction [*Hentschel and Procaccia*, 1983; *Halsey et al.*, 1986]

$$\Gamma_n^q(l_1, l_2, p) = \left(\frac{p^q}{l_1^{\tau(q)}} + \frac{(1-p)^q}{l_2^{\tau(q)}} \right)^n = 1. \quad (1)$$

Namely, after n iterations, τ (q) does not depend on n, we have $\binom{n}{k}$ intervals of width $l = l_1^k\, l_2^{n-k}$, where $k = 1, \ldots, n$, visited with various probabilities. The resulting set of 2^n closed intervals (more and more narrow segments of various widths and probabilities) for $n \to \infty$ becomes the weighted two-scale Cantor set.

[11] Here we consider a standard scenario of cascading eddies, each breaking down into two new ones, but not necessarily equal and twice smaller. In particular, space filling turbulence could be recovered for $l_1 + l_2 = 1$ [*Burlaga et al.*, 1993]. Naturally, in the inertial region of the system of size L, $\eta \ll l \ll L = 1$ (normalized), the energy is not allowed to be dissipated directly, assuming $p_1 + p_2 = 1$, until the Kolmogorov scale η is reached. However, in this range at each nth step of the binomial multiplicative process, the flux of kinetic energy density ε transferred to smaller eddies (energy transfer rate) could be divided into nonequal fractions p and $1-p$, as schematically shown in Figure 1 [cf. *Meneveau and Sreenivasan*, 1987].

3. Solar Wind Data

[12] We have already analyzed the Helios 2 data using plasma parameters measured in situ in the inner heliosphere [*Schwenn*, 1990] for testing of the solar wind attractor [*Macek*, 2007]. The x velocity (mainly radial) component of the plasma flow, υ_x, has already been investigated by *Macek* [1998, 2002, 2003] and *Macek and Redaelli* [2000]. The Alfvénic fluctuations with longer (2-day) samples have been studied by *Macek* [2006a, 2007] and *Macek et al.* [2005, 2006]. To study turbulence cascade, *Macek and Szczepaniak* [2008] have selected 4-day time intervals of υ_x samples in 1976 (solar minimum) for both slow and fast solar wind streams measured at various distances from the Sun. The results for ACE data at 1 AU and dependence on solar cycle is discussed by *Szczepaniak and Macek* [2008]. Here we would like to test asymmetry of the multifractal scaling for the wealth of data provided by another space mission. Namely, we analyze time series of velocities of the solar wind measured by Voyager 2 at various distances from the Sun, 2.5, 25, and 50 AU. Here we have selected even longer (13-day) time intervals of υ_x samples, each of 2^{11} data points, interpolated with sampling time of 192 s (large gaps longer than 10^3 s are not taken into consideration) for both slow and fast solar wind streams during the following solar minima: 1978, 1987–1988, and 1996–1997.

4. Methods of Data Analysis

4.1. Generalized Dimensions

[13] The generalized dimensions D_q as a function of index q [e.g., *Grassberger*, 1983; *Grassberger and Procaccia*, 1983; *Hentschel and Procaccia*, 1983; *Halsey et al.*, 1986] are important characteristics of complex dynamical systems; they quantify multifractality of a given system [*Ott*, 1993]. In the case of turbulence cascade these generalized measures are related to inhomogeneity with which the energy is distributed between different eddies [*Meneveau and Sreenivasan*, 1991]. In this way they provide information about dynamics of multiplicative process of cascading eddies. Here high positive values of $q > 1$ emphasize regions of intense energy transfer rate, while negative values of q accentuate low-transfer-rate regions [cf. *Chhabra et al.*, 1989].

[14] Let us now consider the generalized weighted Cantor set, as shown in Figure 1, where the probability of providing energy for one eddy of size l_1 is p (say, $p \leq 1/2$), and for the other eddy of size l_2 is $1 - p$. For any q in equation (1) one obtains $D_q = \tau(q)/(q-1)$ by solving numerically the following transcendental equation [e.g., *Ott*, 1993]

$$\frac{p^q}{l_1^{\tau(q)}} + \frac{(1-p)^q}{l_2^{\tau(q)}} = 1. \quad (2)$$

4.2. Turbulence Scaling

[15] In the inertial range the averaged standard q-order ($q > 0$) structure function is scaling as

$$S_u^q(l) = \langle |u(x+l) - u(x)|^q \rangle_{\mathrm{av}} \propto l^{\xi(q)}, \quad (3)$$

where $u(x)$ and $u(x + l)$ are velocity components parallel to the longitudinal direction separated from a position x by a distance l. The existence of an inertial range for the experimental data is discussed elsewhere [e.g., *Carbone*, 1994; *Horbury et al.*, 1997; *Szczepaniak and Macek*, 2008]. As is usual, the temporal scales can be interpreted as the spatial scales, $x = \upsilon_{\mathrm{sw}}t$, where υ_{sw} is the average solar wind speed (Taylor's hypothesis). The transfer rate of the energy flux $\varepsilon(l)$ is widely estimated by

$$\varepsilon(l) \sim \frac{|u(x+l) - u(x)|^3}{l}. \quad (4)$$

Therefore, to each ith eddy of size l in turbulence cascade ($i = 1, \ldots, N = 2^n$) we associate a probability measure defined by

$$p_i(l) = \frac{\varepsilon_i(l)}{\sum_{i=1}^N \varepsilon_i(l)}. \quad (5)$$

This quantity can roughly be interpreted as a probability that the energy flux is transferred to an eddy of size $l = \upsilon_{\mathrm{sw}}t$. Here, for simplicity the third moment of structure function of velocity fluctuations in equation (4) is used for estimation of this measure [*Marsch et al.*, 1996]. Recently, limitations of this approximation are discussed by *Vasquez et al.* [2007] using power spectra, and hydromagnetic generalization of this approximation for the Alfvénic fluctuations is considered by *Sorriso-Valvo et al.* [2007].

[16] The multifractal measure $\varepsilon(t)/\langle \varepsilon_L(t) \rangle_{\mathrm{av}}$ (normalized) on the unit interval of equation (5) for the usual one-scale p model [*Meneveau and Sreenivasan*, 1987; *Mandelbrot*,

1989] and the generalized two-scale cascade model is shown in Figure 2 of *Macek and Szczepaniak* [2008]. It is worth noting that intermittent pulses are much stronger for the model with two different scaling parameters. In particular, for non space-filling turbulence, $l_1 + l_2 < 1$, one still could have a multifractal cascade, even for unweighted (equal) energy transfer, $p = 0.5$. Only for $l_1 = l_2 = 0.5$ and $p = 0.5$ there is no multifractality.

[17] Now, for a continuous index $-\infty < q < \infty$ using a q-order total probability measure

$$I(q,l) \equiv \sum_{i=1}^{N} p_i^q(l) \tag{6}$$

and a q-order generalized information $H(q, l)$ (corresponding to Renyi's entropy) defined by *Grassberger* [1983]

$$H(q,l) \equiv -\log I(q,l) = -\log \sum_{i=1}^{N} p_i^q(l) \tag{7}$$

one obtains the usual q-order generalized dimensions [*Hentschel and Procaccia*, 1983] $D_q \equiv \tau(q)/(q-1)$, where

$$\tau(q) = \lim_{l \to 0} \frac{H(q,l)}{\log(1/l)}. \tag{8}$$

[18] From equations (3) to (8) for d-dimensional cascade model one can have [*Tsang et al.*, 2005]

$$\tau(q) = d(q-1) + \xi(3q) - q\xi(3). \tag{9}$$

Admittedly, the structure function scaling exponent $\xi(q)$ is easier to measure experimentally than the spectrum of dimensions $D_q \equiv \tau(q)/(q-1)$ in equation (8), which is easier to interpret theoretically; see equation (1). Surely, as seen from equation (9), both have the same information about multifractality, at least for $q \geq 0$. However, because we are also interested in negative q, it is more convenient to use dimensions instead of structure functions for looking into the multifractality.

[19] More precisely, one may distinguish a probability measure from its geometrical support, which may or may not have fractal geometry. Then, if the measure has different fractal dimensions on different parts of the support, the measure is multifractal [*Mandelbrot*, 1989]. Therefore, to each probability measure p_i of ith eddy of size l in turbulence cascade we associate a singularity index $\alpha_i(l) \equiv \log p_i(l)/\log l$ [*Halsey et al.*, 1986]. Similarly, using equation (6), we define a one-parameter q family of (normalized) generalized pseudoprobability measures [*Chhabra and Jensen*, 1989; *Chhabra et al.*, 1989]

$$\mu_i(q,l) \equiv \frac{p_i^q(l)}{I(q,l)}. \tag{10}$$

Now, with an associated fractal dimension index $f_i(q, l) \equiv \log \mu_i(q, l)/\log l$ for a given q the multifractal singularity spectrum of dimensions is defined directly as the averages taken with respect to the measure $\mu(q, l)$ in equation (10) denoted from here on by $\langle \ldots \rangle$ (skipping a subscript av)

$$f(q) \equiv \lim_{l \to 0} \sum_{i=1}^{N} \mu_i(q,l) f_i(q,l) = \langle f(q) \rangle \tag{11}$$

and the corresponding average value of the singularity strength is given by *Chhabra and Jensen* [1989]:

$$\alpha(q) \equiv \lim_{l \to 0} \sum_{i=1}^{N} \mu_i(q,l)\alpha_i(l) = \langle \alpha(q) \rangle. \tag{12}$$

[20] Hence, by using a q-order mixed Shannon information entropy

$$S(q,l) = -\sum_{i=1}^{N} \mu_i(q,l) \log p_i(l) \tag{13}$$

we obtain the singularity strength as a function of q

$$\alpha(q) = \lim_{l \to 0} \frac{S(q,l)}{\log(1/l)} = \lim_{l \to 0} \frac{\langle \log p_i(l) \rangle}{\log(l)}, \tag{14}$$

Similarly, by using the q-order generalized Shannon entropy

$$K(q,l) = -\sum_{i=1}^{N} \mu_i(q,l) \log \mu_i(q,l) \tag{15}$$

we obtain directly the singularity spectrum as a function of q

$$f(q) = \lim_{l \to 0} \frac{K(q,l)}{\log(1/l)} = \lim_{l \to 0} \frac{\langle \log \mu_i(q,l) \rangle}{\log(l)}. \tag{16}$$

One can easily verify that the multifractal singularity spectrum $f(\alpha)$ as a function of α satisfies the following Legendre transformation [*Halsey et al.*, 1986; *Jensen et al.*, 1987]:

$$\alpha(q) = \frac{d\tau(q)}{dq}, \qquad f(\alpha) = q\alpha(q) - \tau(q). \tag{17}$$

5. Results and Discussion

[21] First, in order to estimate the multifractal spectrum for solar wind turbulence, we should calculate the multifractal measure given in equation (5). The values obtained using data of the velocity components $u = v_x$ measured by Voyager 2 during solar minimum (1978, 1987–1988, and 1996–1997) at 2.5, 25, and 50 AU are presented for the slow (Figures 2a, 2c, and 2e) and fast (Figures 2b, 2d, and 2f) solar wind, correspondingly.

[22] Second, for a given q, we calculate the generalized q-order total probability measure $I(q, l)$ of equation (6) as a function of various scales l that cover turbulence cascade [cf. *Macek and Szczepaniak*, 2008, equation (2)]. On a small scale l in the scaling region one should have, according to equations (6) to (8), $I(q, l) \propto l^{\tau(q)}$, where $\tau(q)$ is an approximation of the ideal limit $l \to 0$ solution of equation (8) [e.g., *Macek et al.*, 2005, equation (1)]. Equivalently, writing $I(q, l) = \Sigma p_i (p_i)^{q-1}$ as a usual weighted average of $\langle (p_i)^{q-1} \rangle_{av}$, one can associate bulk with the generalized average probability per cascading eddies

$$\bar{\mu}(q,l) \equiv \sqrt[q-1]{\langle (p_i)^{q-1} \rangle_{av}}, \tag{18}$$

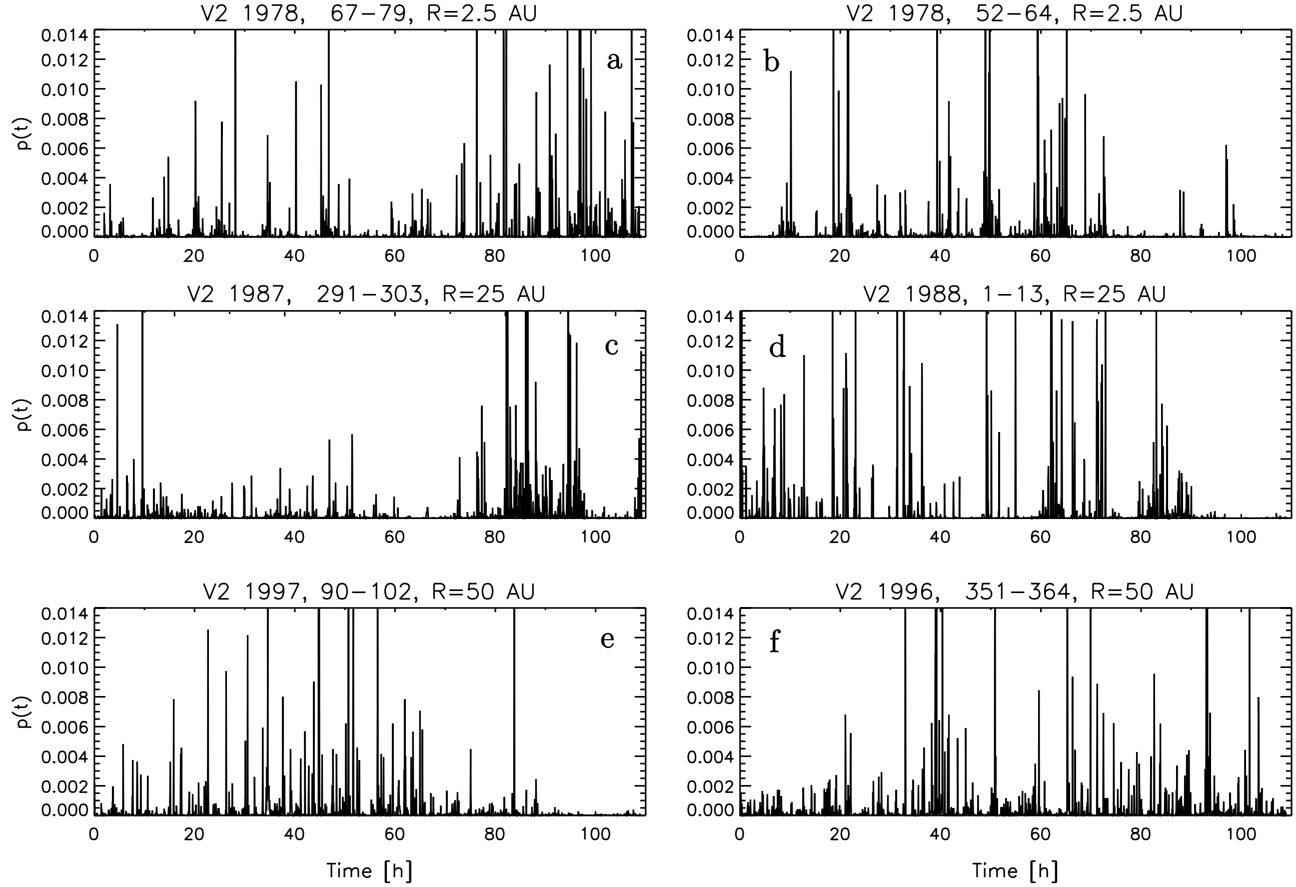

Figure 2. The normalized transfer rate of the energy flux $p(t) = \varepsilon_i(t)/\sum\varepsilon_i(t)$ obtained using data of the v_x velocity components measured by Voyager 2 during solar minimum (1978, 1987–1988, 1996–1997) at 2.5, 25, and 50 AU for the (a, c, and e) slow and (b, d, and f) fast solar wind, correspondingly.

and identify D_q as a scaling of bulk with size l,

$$\overline{\mu}(q, l) \propto l^{D_q}. \tag{19}$$

Hence, the slopes of the logarithm of $\overline{\mu}(q, l)$ of equation (19) versus log l (normalized) provides

$$D_q(l) = \frac{\log \overline{\mu}(q, l)}{\log l}. \tag{20}$$

[23] In Figure 3 we have depicted some of the values of the generalized average probability, $\log\overline{\mu}(q, l)$, for the following values of q: 6, 4, 2, 1, 0, −1, −2, using data measured by Voyager 2 during solar minimum (1978) at 2.5 AU (diamonds) for the slow (Figure 3a) and fast (Figure 3b) solar wind, correspondingly. If we can fit these values to a straight line in a scaling region, $x_{\min} < \log l < x_{\max}$, this slope on the logarithmic scales can be identified with the requested generalized dimension. Then, the average slope is taken as D_q, with the statistical errors taken as half the difference of the maximum and minimum slopes (obtained using unweighted least squares fitting) over the scaling range [cf. *Macek*, 1998, 2006a, 2007; *Macek et al.*, 2005, 2006]. We have also verified that the slopes in the scaling region are not sensitive to the number of points used [see, e.g., *Eckmann and Ruelle*, 1992].

[24] The results for the generalized dimensions D_q as a function of q, calculated from equation (20) using the Helios 2 data and compared with those obtained from equation (2) for solar wind turbulence for the slow and fast solar wind streams at distances of 0.3 AU and 0.97 AU, correspondingly, are presented in Figures 3a, 3b, 3c and 3d of the paper by *Macek and Szczepaniak* [2008]. The related singularity spectra $f(\alpha)$ as a function of α are presented by [*Macek*, 2009, Figure 2]. In particular, in agreement with other studies, we confirm the universal shape of the multifractal spectrum as noticed, e.g., by *Burlaga* [2001].

[25] In this paper our calculations are repeated for velocity fluctuations measured by Voyager 2 spacecraft at various distances of 2.5, 25, and 50 AU. Namely, using the slopes in Figure 3 the corresponding results for the generalized dimensions D_q as a function of q in equation (20), with the statistical errors of the average slopes over the scaling range are shown in Figure 4. In addition, in Figures 5 and 6 the generalized average logarithmic probability and pseudoprobability measures of cascading eddies $\langle \log_{10} p_i(l) \rangle$ and $\langle \log_{10} \mu_i(q, l) \rangle$ versus $\log_{10} l$, as given in equations (5) and (10), are demonstrated for some positive and negative values of q: 2, 1, 0, −1, −2, as obtained using data

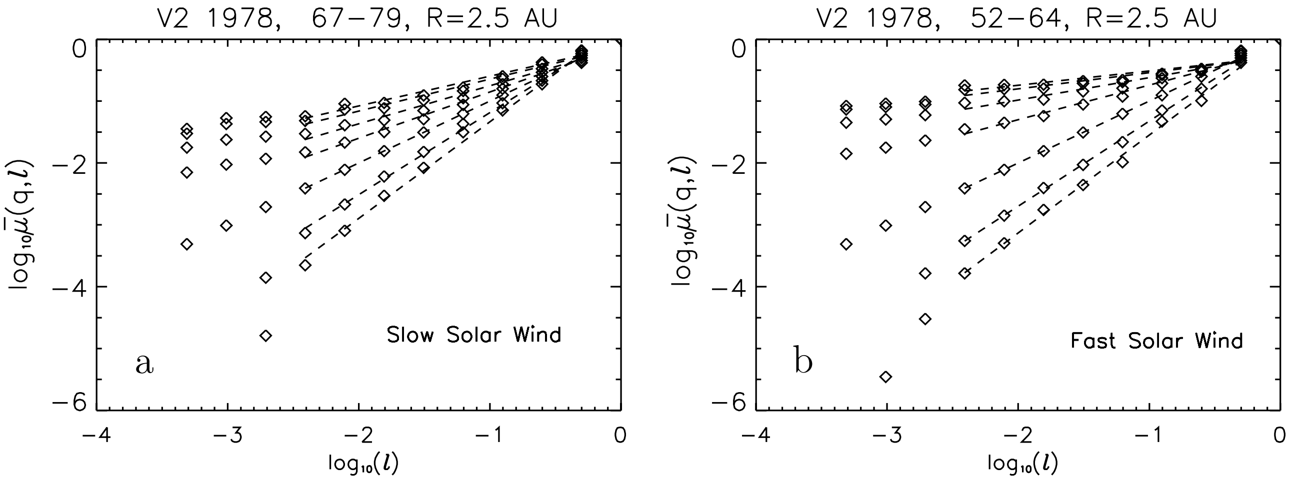

Figure 3. Plots of the generalized average probability of cascading eddies $\log_{10}\overline{\mu}(q, l)$ versus $\log_{10} l$ for the following values of q: 6, 4, 2, 1, 0, -1, -2. These results are obtained using data measured by Voyager 2 during solar minimum (1978) at 2.5 AU (diamonds) for the (a) slow and (b) fast solar wind, correspondingly.

measured by Voyager 2 during solar minimum (1978) at 2.5 AU (diamonds) for the slow (Figures 5a and 6a) and fast (Figures 5b and 6b) solar wind, correspondingly. Please note that here the averages are taken with respect to the generalized measure $\mu(q, l)$ in equation (10).

[26] According to equations (13) and (15) these average quantities are simply related to the generalized Shannon information entropies. These results are obtained using data of the v_x velocity components measured by Voyager 2 during solar minimum (1978, 1987–1988, 1996–1997) at various distance from the Sun: 2.5, 25, and 50 AU (diamonds) for the slow (Figures 7a, 7c, and 7e) and fast (Figures 7b, 7d, and 7f) solar wind, correspondingly. In this way, the singularity spectra $f(\alpha)$ are obtained directly from the data as a function of α as given by equations (14) and (16) and the results are presented in Figure 7. In fact, both values of D_q and $f(\alpha)$ for one-dimensional turbulence, $d = 1$, are calculated using the radial velocity components $u = v_x$ (in time domain) in equation (4) [cf. *Macek and Szczepaniak*, 2008, Figure 3]. Admittedly, as is well known, for $q < 0$ we have some basic statistical problems [*Macek*, 2006a, 2007]. Nevertheless, in spite of large statistical errors in Figures 4 and 7, especially for $q < 0$, the multifractal character of the measure can still clearly be discerned. Therefore, one can confirm that the spectrum of dimensions still exhibits the multifractal structure of the solar wind in the outer heliosphere.

[27] For $q \geq 0$ these results agree with the usual one-scale p model fitted to the generalized dimensions as obtained analytically using $l_1 = l_2 = 0.5$ in equation (2) and the corresponding value of the parameter $p \simeq 0.24$, 0.19, and 0.14 and 0.13, 0.22, and 0.19 during solar minimum (1978, 1987–1988, and 1996–1997) for the slow (Figures 7a, 7c, and 7e) and fast (Figures 7b, 7d, and 7f) solar wind streams, correspondingly, as shown by dashed lines. The values of parameter p are related to the usual models, which are based on the p model of turbulence [e.g., *Meneveau and Sreenivasan*, 1987]. These values of p obtained here are roughly consistent with the fitted value in the literature both for laboratory and the solar wind turbulence, which is in the range $0.1 \leq p \leq 0.3$ [e.g., *Burlaga*, 1991; *Carbone*, 1993; *Carbone and Bruno*, 1996; *Marsch et al.*, 1996]. On the contrary, in general for $q < 0$ (right part of the singularity spectrum in Figure 7) the p model cannot describe the observational results, as already noted by *Marsch et al.* [1996]. Admittedly, a deviation from one-scale multifractal scaling can sometimes be attributed to a suspicion that turbulence under study is not actually fully developed. But we show that the experimental values are consistent with the generalized dimensions obtained numerically from equation (2) for the weighted two-scale Cantor set using an asymmetric scaling, i.e., using unequal scales $l_1 \neq l_2$, as is depicted in Figures 4a–4f and 7a–7f by continuous lines.

[28] Finally, we can take

$$\Delta \equiv \alpha_{\max} - \alpha_{\min} = D_{-\infty} - D_{\infty} = \left| \frac{\log(1-p)}{\log l_2} - \frac{\log(p)}{\log l_1} \right| \quad (21)$$

as the degree of multifractality [see, e.g., *Macek*, 2006a, 2007]. Moreover, using the value of the strength of singularity α_0 at which the singularity spectrum has its maximum $f(\alpha_0) = 1$ we define a measure of asymmetry by

$$A \equiv \frac{\alpha_0 - \alpha_{\min}}{\alpha_{\max} - \alpha_0}. \quad (22)$$

We see from Table 1 that the obtained values of Δ obtained from equation (21) for the solar wind in the outer heliosphere are somewhat different for slow and fast streams (Figure 4). For the fast wind (not very far away from the Sun) D_q fells more steeply with q than for the slow wind, and therefore one can say that the degree of multifractality is larger for the fast wind.

[29] In Figure 7 we show the corresponding singularity spectra with the values of A of equation (22) also presented in Table 1. It worth noting that the multifractal scaling is often rather asymmetric, and we observe the evolution of multifractal scaling (intermittency) in the outer heliosphere

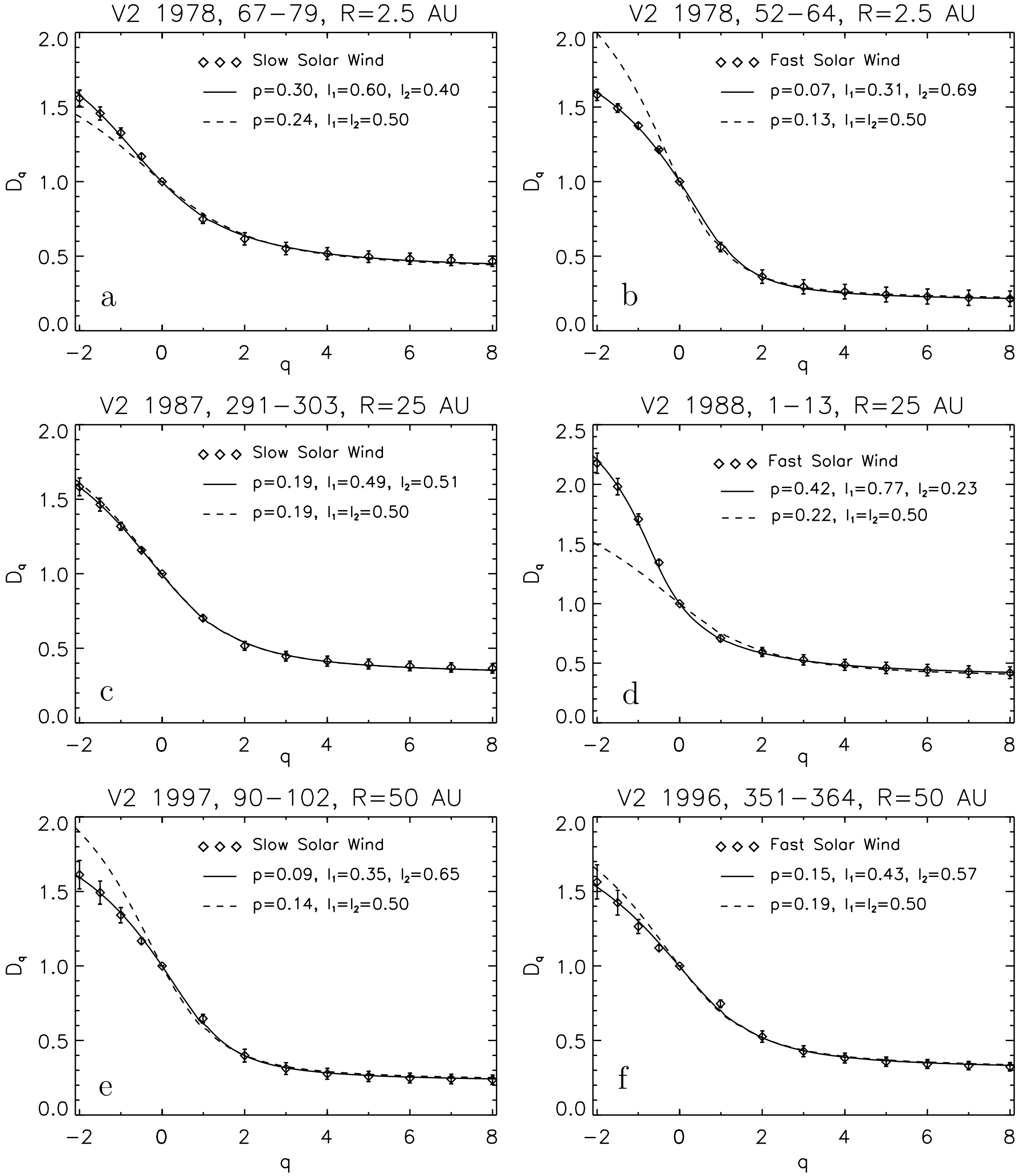

Figure 4. The generalized dimensions D_q calculated for the one-scale p model (dashed lines) and the generalized two-scale (continuous lines) models with parameters fitted to the multifractal measure $\mu(q, l)$ using data measured by Voyager 2 during solar minimum (1978, 1987–1988, 1996–1997) at 2.5, 25, and 50 AU (diamonds) for the (a, c, and e) slow and (b, d, and f) fast solar wind, correspondingly.

[cf. *Burlaga*, 1991; *Burlaga et al.*, 2003; *Burlaga*, 2004], as also noticed in the inner heliosphere, e.g., by *Bruno et al.* [2003]. In particular, we see that the degree of multifractality for the solar wind during solar minimum is usually somewhat greater for fast streams velocity fluctuations than that for the slow streams. One can also say that in the fast streams during solar minimum the scaling is more asymmetric than that for the slow wind. In fact, the fast wind can exhibit strong asymmetric scaling. Moreover, this degree of multifractality and degree of asymmetry are somehow correlated with the heliospheric distance. It seems that the degree of asymmetry for the slow wind is only weakly correlated with the heliospheric distance, but more strongly correlated for the fast wind (A changes from 0.7 to 1.5); only the slow wind near solar minimum during some periods (maybe not very far from the Sun) may exhibit

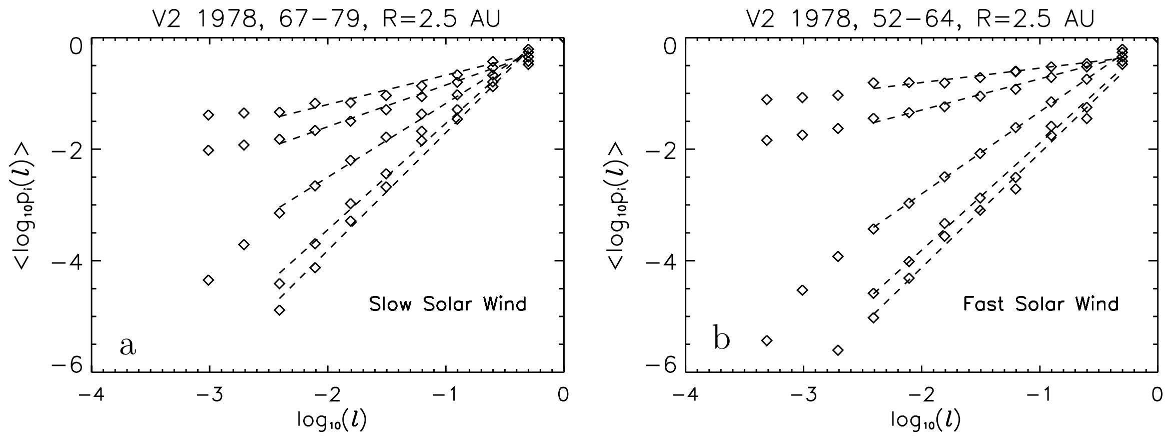

Figure 5. Plots of the generalized average logarithmic probability of cascading eddies $\langle \log_{10} p_i(l) \rangle$ versus $\log_{10} l$ for the following values of q: 2, 1, 0, −1, −2. These results are obtained using data measured by Voyager 2 during solar minimum (1978) at 2.5 AU (diamonds) for the (a) slow and (b) fast solar wind, correspondingly.

roughly symmetric scaling, $A \sim 1$, and one-scale Cantor set model applies.

[30] Moreover, it is clear that the multifractal spectrum of the solar wind is only roughly consistent with that for the multifractal measure of the self-similar weighted symmetric one-scale weighted Cantor set only for $q \geq 0$, as also seen from the standard structure function analysis. Only for the slow wind during solar minimum both models give the similar results, and only in this case we have the symmetric singularity spectrum. On the other hand, all these spectra are in a very good agreement with the two-scale asymmetric weighted Cantor set schematically shown in Figure 1 for both positive and negative q. Obviously, taking two different scales for eddies in the cascade, one obtains a more general situation than in the usual p model of *Meneveau and Sreenivasan* [1987] for fully developed turbulence, especially for an asymmetric scaling, $l_1 \neq l_2$. Hence, we hope that this generalized model will be a useful tool for analysis of intermittent turbulence in space plasmas.

6. Conclusions

[31] We have studied the inhomogeneous rate of the transfer of the energy flux indicating multifractal and intermittent behavior of solar wind turbulence in the outer heliosphere. In particular, we have demonstrated that for the model with two different scaling parameters a much better agreement with the real data is obtained, especially for $q < 0$. By investigating the Voyager data we confirm that the degree of multifractality of the solar wind in the outer heliosphere is different for slow and fast streams. Namely, our analysis show that the degree of multifractality for the solar wind during solar minimum is somewhat greater for fast streams velocity fluctuations than that for the slow

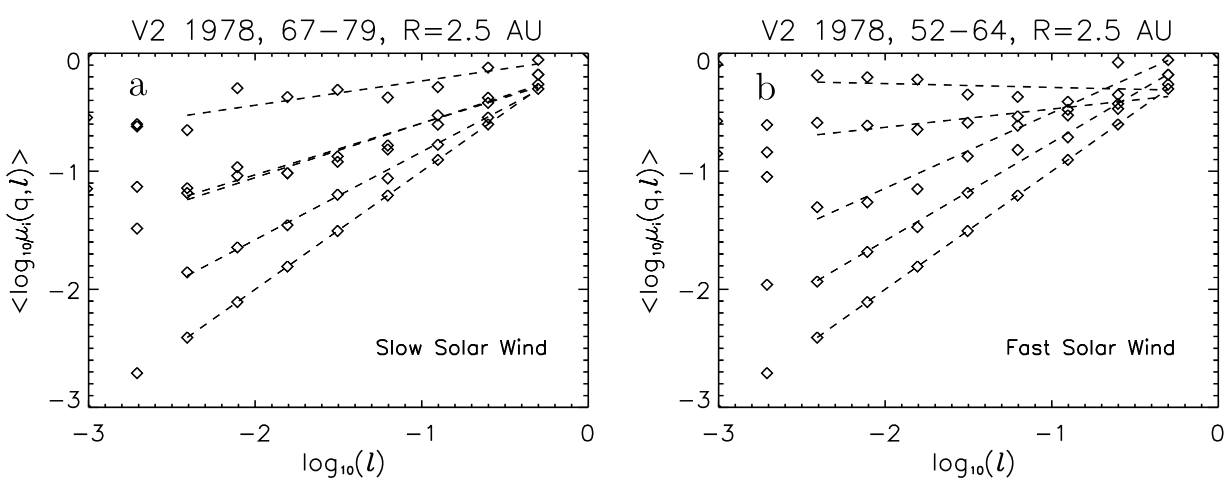

Figure 6. Plots of the generalized average logarithmic pseudoprobability measure of cascading eddies $\langle \log_{10} \mu_i(q, l) \rangle$ versus $\log_{10} l$ for the following values of q: 2, 1, 0, −1, −2. These results are obtained using data measured by Voyager 2 during solar minimum (1978) at 2.5 AU (diamonds) for the (a) slow and (b) fast solar wind, correspondingly.

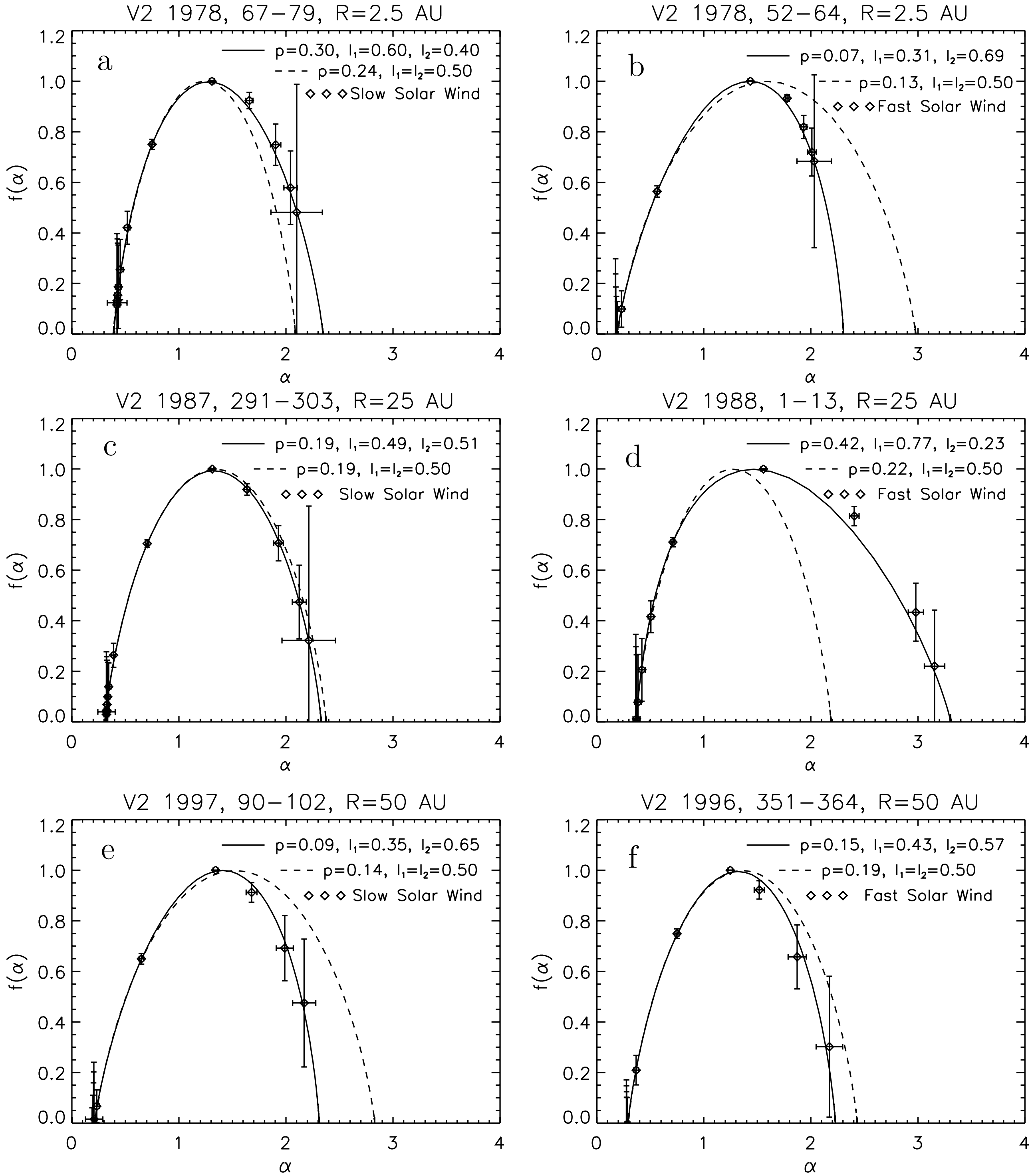

Figure 7. The singularity spectrum $f(\alpha)$ calculated for the one-scale p model (dashed lines) and the generalized two-scale (continuous lines) models with parameters fitted to the multifractal measure $\mu(q, l)$ using data measured by Voyager 2 during solar minimum (1978, 1987–1988, 1996–1997) at 2.5, 25, and 50 AU (diamonds) for the (a, c, and e) slow and (b, d, and f) fast solar wind, correspondingly.

streams; the generalized dimensions varies more with the index q. It is worth noting that the multifractal scaling is often rather asymmetric. In particular, the fast wind during solar minimum exhibits strong asymmetric scaling. Moreover, both the degree of multifractality and degree of asymmetry are correlated with the heliospheric distance and we observe the evolution of multifractal scaling in the outer heliosphere [cf. *Burlaga*, 1991; *Burlaga et al.*, 2003; *Burlaga*, 2004].

Table 1. Degree of Multifractality Δ and Asymmetry A for Solar Wind Data in the Outer Heliosphere During Solar Minimum

Heliospheric Distance (Year)	Slow Solar Wind	Fast Solar Wind
2.5 AU (1978)	Δ = 1.95, A = 0.91	Δ = 2.12, A = 1.54
25 AU (1987–1988)	Δ = 2.02, A = 0.98	Δ = 2.93, A = 0.66
50 AU (1996–1997)	Δ = 2.10, A = 1.14	Δ = 1.94, A = 0.95

[32] Basically, the generalized dimensions for solar wind are consistent with the generalized p model for both positive and negative q, but rather with different scaling parameters for sizes of eddies, while the usual p model can only reproduce the spectrum for $q \geq 0$. Thus, we also confirm the utility of the model introduced by *Burlaga et al.* [1993], using a different data set. In general, the proposed generalized two-scale weighted Cantor set model should also be valid for non space filling turbulence. Hence, we hope that our new more general asymmetric multifractal model could shed light on the nature of turbulence and we therefore propose this model as a useful tool for analysis of intermittent turbulence in various environments.

[33] **Acknowledgments.** We would like to thank the plasma instruments team of Voyager mission, NASA National Space Science Data, and the Space Science Data Facility for providing velocity data. The authors thank the referee for his critical comments intended to improve our paper. This work has been supported by the Ministry of Science and Higher Education (MNiSW) through grant NN 202 4127 33. One of us (W.M.M.) would also like to acknowledge a support from the Japanese Society for the Promotion of Science.

[34] Amitava Bhattacharjee thanks the reviewer for their assistance in evaluating this paper.

References

Bruno, R., and V. Carbone (2005), The solar wind as a turbulence laboratory, *Living Rev. Sol. Phys.*, *2*, lrsp-2005-4. (Available at http://www.livingreviews.org/lrsp-2005-4.)

Bruno, R., V. Carbone, P. Veltri, E. Pietropaolo, and B. Bavassano (2001), Identifying intermittency events in the solar wind, *Planet. Space Sci.*, *49*, 1201–1210.

Bruno, R., V. Carbone, L. Sorriso-Valvo, and B. Bavassano (2003), Radial evolution of solar wind intermittency in the inner heliosphere, *J. Geophys. Res.*, *108*(A3), 1130, doi:10.1029/2002JA009615.

Burlaga, L. F. (1991), Multifractal structure of the interplanetary magnetic field: Voyager 2 observations near 25 AU, 1987–1988, *Geophys. Res. Lett.*, *18*, 69–72.

Burlaga, L. F. (2001), Lognormal and multifractal distributions of the heliospheric magnetic field, *J. Geophys. Res.*, *106*(A8), 15,917–15,927.

Burlaga, L. F. (2004), Multifractal structure of the large-scale heliospheric magnetic field strength fluctuations near 85 AU, *Nonlinear Processes Geophys.*, *11*, 441–445. (Available at http://www.nonlin-processes-geophys.net/11/441/2004/)

Burlaga, L. F., J. Perko, and J. Pirraglia (1993), Cosmic-ray modulation, merged interaction regions and multifractals, *Astrophys. J.*, *407*, 347–358, doi:10.1086/172517.

Burlaga, L. F., C. Wang, and N. F. Ness (2003), A model and observations of the multifractal spectrum of the heliospheric magnetic field strength fluctuations near 40 AU, *Geophys. Res. Lett.*, *30*(10), 1543, doi:10.1029/2003GL016903.

Carbone, V. (1993), Cascade model for intermittency in fully developed magnetohydrodynamic turbulence, *Phys. Rev. Lett.*, *71*, 1546–1548, doi:10.1103/PhysRevLett.71.1546.

Carbone, V. (1994), Scaling exponents of the velocity structure functions in the interplanetary medium, *Ann. Geophys.*, *12*, 585–590.

Carbone, V., and R. Bruno (1996), Cancellation exponents and multifractal scaling laws in the solar wind magnetohydrodynamic turbulence, *Ann. Geophys.*, *14*, 777–785.

Chhabra, A., and R. V. Jensen (1989), Direct determination of the $f(\alpha)$ singularity spectrum, *Phys. Rev. Lett.*, *62*, 1327–1330, doi:10.1103/PhysRevLett.62.1327.

Chhabra, A. B., C. Meneveau, R. V. Jensen, and K. R. Sreenivasan (1989), Direct determination of the $f(\alpha)$ singularity spectrum and its application to fully developed turbulence, *Phys. Rev. A*, *40*, 5284–5294, doi:10.1103/PhysRevA.40.5284.

Eckmann, J.-P., and D. Ruelle (1992), Fundamental limitations for estimating dimensions and Lyapunov exponents in dynamical systems, *Physica D*, *56*, 185–187, doi:10.1016/0167-2789(92)90023-G.

Falconer, K. (1990), *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley, New York.

Frisch, U. (1995), *Turbulence: The Legacy of A. N. Kolmogorov*, Cambridge Univ. Press, Cambridge, U.K.

Grassberger, P. (1983), Generalized dimensions of strange attractors, *Phys. Lett. A*, *97*, 227–230, doi:10.1016/0375-9601(83)90753-3.

Grassberger, P., and I. Procaccia (1983), Measuring the strangeness of strange attractors, *Physica D*, *9*, 189–208, doi:10.1016/0167-2789(83)90298-1.

Halsey, T. C., M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman (1986), Fractal measures and their singularities: The characterization of strange sets, *Phys. Rev. A*, *33*, 1141–1151, doi:10.1103/PhysRevA.33.1141.

Hentschel, H. G. E., and I. Procaccia (1983), The infinite number of generalized dimensions of fractals and strange attractors, *Physica D*, *8*, 435–444, doi:10.1016/0167-2789(83)90235-X.

Hnat, B., S. C. Chapman, and G. Rowlands (2003), Intermittency, scaling, and the Fokker-Planck approach to fluctuations of the solar wind bulk plasma parameters as seen by the WIND spacecraft, *Phys. Rev. E*, *67*, 056404, doi:10.1103/PhysRevE.67.056404.

Hnat, B., S. C. Chapman, K. Kiyani, G. Rowlands, and N. W. Watkins (2007), On the fractal nature of the magnetic field energy density in the solar wind, *Geophys. Res. Lett.*, *34*, L15108, doi:10.1029/2007GL029531.

Horbury, T. S., and A. Balogh (2001), Evolution of magnetic field fluctuations in high-speed solar wind streams: Ulysses and Helios observations, *J. Geophys. Res.*, *106*, 15,929–15,940.

Horbury, T. S., A. Balogh, R. J. Forsyth, and E. J. Smith (1997), Ulysses observations of intermittent heliospheric turbulence, *Adv. Space Res.*, *19*(6), 847–850, doi:10.1016/S0273-1177(97)00290-1.

Jensen, M. H., L. P. Kadanoff, and I. Procaccia (1987), Scaling structure and thermodynamics of strange sets, *Phys. Rev. A*, *36*, 1409–1420, doi:10.1103/PhysRevA.36.1409.

Kiyani, K., S. C. Chapman, B. Hnat, and R. M. Nicol (2007), Self-similar signature of the active solar corona within the inertial range of solar-wind turbulence, *Phys. Res. Lett.*, *98*, 211101, doi:10.1103/PhysRevLett98.211101.

Macek, W. M (1998), Testing for an attractor in the solar wind flow, *Physica D*, *122*, 254–264, doi:10.1016/S0167-2789(98)00098-0.

Macek, W. M. (2002), Multifractality and chaos in the solar wind, in *Experimental Chaos*, edited by S. Boccaletti et al., *AIP Conf. Proc.*, *622*, 74–79, doi:10.1063/1.1487522.

Macek, W. M. (2003), The multifractal spectrum for the solar wind flow, in *Solar Wind Ten*, edited by M. Velli, R. Bruno, and F. Malara, *AIP Conf. Proc.*, *679*, 530–533, doi:10.1063/1.1618651.

Macek, W. M. (2006a), Modeling multifractality of the solar wind, *Space Sci. Rev.*, *122*, 329–337, doi:10.1007/s11214-006-8185-z.

Macek, W. M. (2006b), Multifractal solar wind, *Academia*, *3*(11), 16–18.

Macek, W. M. (2007), Multifractality and intermittency in the solar wind, *Nonlinear Processes Geophys.*, *14*, 695–700. (Available at http://www.nonlin-processes-geophys.net/14/695/2007/npg-14-695-2007.html.)

Macek, W. M. (2009), Chaos and multifractals in the solar wind, *Adv. Space Res.*, in press.

Macek, W. M., and S. Redaelli (2000), Estimation of the entropy of the solar wind flow, *Phys. Rev. E*, *62*, 6496–6504, doi:10.1103/PhysRevE.62.6496.

Macek, W. M., and A. Szczepaniak (2008), Generalized two-scale weighted Cantor set model for solar wind turbulence, *Geophys. Res. Lett.*, *35*, L02108, doi:10.1029/2007GL032263.

Macek, W. M., R. Bruno, and G. Consolini (2005), Generalized dimensions for fluctuations in the solar wind, *Phys. Rev. E*, *72*, 017202, doi:10.1103/PhysRevE.72.017202.

Macek, W. M, R. Bruno, and G. Consolini (2006), Testing for multifractality of the slow solar wind, *Adv. Space Res*, *37*(3), 461–466, doi:10.1016/j.asr.2005.06.057.

Mandelbrot, B. B. (1989), Multifractal measures, especially for the geophysicist, *Pure Appl. Geophys.*, *131*, 5–42, doi:10.1007/BF00874478.

Marsch, E., and C.-Y. Tu (1997), Intermittency, non-Gaussian statistics and fractal scaling of MHD fluctuations in the solar wind, *Nonlinear Processes Geophys.*, *4*, 101–124. (Available at http://www.nonlin-processes-geophys.net/4/101/1997/.)

Marsch, E., C.-Y. Tu, and H. Rosenbauer (1996), Multifractal scaling of the kinetic energy flux in solar wind turbulence, *Ann. Geophys.*, *14*, 259–269.

Meneveau, C., and K. R. Sreenivasan (1987), Simple multifractal cascade model for fully developed turbulence, *Phys. Rev. Lett.*, *59*, 1424–1427, doi:10.1103/PhysRevLett.59.1424.

Meneveau, C., and K. R. Sreenivasan (1991), The multifractal nature of turbulent energy dissipation, *J. Fluid Mech*, *224*, 429–484, doi:10.1017/S0022112091001830.

Ott, E. (1993), *Chaos in Dynamical Systems*, Cambridge Univ. Press, Cambridge, U. K.

Richardson, L. F. (1922), *Weather Prediction by Numerical Process*, Cambridge Univ. Press, Cambridge, U. K.

Schwenn, R. (1990), Large-scale structure of the interplanetary medium, in *Physics of the Inner Heliosphere*, *20*, edited by R. Schwenn and E. Marsch, pp. 99–181, Springer-Verlag, Berlin.

Sorriso-Valvo, L., R. Marino, V. Carbone, F. Lepreti, P. Veltri, A. Noullez, R. Bruno, B. Bavassano, and E. Pietropaolo (2007), Observation of inertial energy cascade in interplanetary space plasma, *Phys. Rev. Lett.*, *99*, 115001, doi:10.1103/PhysRevLett.99.115001.

Szczepaniak, A., and W. M. Macek (2008), Asymmetric multifractal model for solar wind intermittent turbulence, *Nonlinear Processes Geophys.*, *15*, 615–620. (Available at http://www.nonlin-processes-geophys.net/15/615/2008/.)

Tsang, Y.-K., E. Ott, T. M. Antonsen Jr., and P. N. Guzdar (2005), Intermittency in two-dimensional turbulence with drag, *Phys. Rev. E*, *71*, 066313, doi:10.1103/PhysRevE.71.066313.

Vasquez, B. J., C. W. Smith, K. Hamilton, B. T. MacBride, and R. J. Leamon (2007), Evaluation of the turbulent energy cascade rates from the upper inertial range in the solar wind at 1 AU, *J. Geophys. Res.*, *112*, A07101, doi:10.1029/2007JA012305.

W. M. Macek and A. Wawrzaszek, Space Research Centre, Polish Academy of Sciences, Bartycka 18 A, 00-716 Warszawa, Poland. (macek@cbk.waw.pl)