Time Series Multifractal Analysis of Turbulent Magnetized Plasmas in the Solar System

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Abstract. The aim of this paper is to report on the new developments of time series analysis in turbulence using multifractals. To quantify scaling of turbulence we have proposed a generalized two-scale weighted Cantor set model with two different rescaling parameters. By using wealth of data provided by various space missions we have applied this model to intermittent multifractal turbulence in the solar wind magnetized plasma in the inner and the outer heliosphere at the ecliptic and at high heliospheric latitudes and even in the heliosheath, beyond the heliospheric termination shock. It appears that the degree of multifractality falls steadily with the distance from the Sun and is modulated by the solar activity. Here we extend our analysis further in the heliosheath ahead of the heliopause using recent Voyager 1 magnetic field data. We hope that our generalized multifractal model will be a useful tool for analysis of intermittent turbulence in the Solar System plasmas. We thus believe that multifractal time series analysis of various complex environments can shed light on the nature of turbulence.

Keywords: time series, multifractal, turbulence, solar wind, plasma

1 Introduction

We have used time series analysis of solar wind plasma with frozen-in magnetic field measured during space missions onboard various spacecraft, such as Helios, Advanced Composition Explorer, Ulysses, and Voyager, exploring different regions of the heliosphere during solar minimum and maximum. At present we focus on the fluctuations of the interplanetary magnetic field strength observed by both Voyager 1 and 2 spacecraft in the outer heliosphere and in the heliosheath, i.e., after crossing the heliospheric termination heliospheric shock (in 2004 and 2007, correspondingly), and even now ahead of the heliopause, which is the last boundary separating the heliospheric plasma from the local interstellar plasma. In fact, on 12 September 2013 NASA announced that Voyager 1 crossed the heliopause in 2012 at heliospheric distances of 122 astronomical units (AU) and for the first time in the human history has entered the interstellar medium.

To quantify scaling of solar wind turbulence, we have proposed a generalized two-scale weighted Cantor set model with two different rescaling parameters [1, 2]. Using this phenomenological model, we have shown that this complex nonlinear solar system may exhibit multifractal scaling of plasma characteristic parameters resulting in multifractal turbulence. We have investigated the singularity multifractal spectrum and showed that our model can explain the observed spectrum, which is often asymmetric in the outer heliosphere [3], in contrast to the nearly symmetric spectrum in the heliosheath [4]. We have noticed a change of the asymmetry of the spectrum at the termination shock. We have also shown that the degree of multifractality falls steadily with the distance from the Sun (toward a monofractal behaviour) and is modulated by the solar activity, resulting in the evolution of the Solar System plasma [4, 5].

In this paper we extend our analysis further in the heliosheath ahead of the heliopause. We therefore hope that the proposed generalized multifractal model will be a useful tool for analysis of intermittent turbulence in the Solar System plasmas. We thus believe that multifractal time series analysis of various complex environments can shed light on the nature of turbulence.

2 Fractals and Multifractals

The basic concepts of fractal sets are explained in several textbooks, see e.g. [6, 7]. We only remind here that a fractal is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. For example, strange attractors are often fractal sets, which exhibits a hidden order within chaos. Fractals are generally *self-similar* and independent of scale (generally with a particular fractal dimension). On the other hand, a multifractal is a more complex object that demonstrate various self-similarities, described by a multifractal spectrum of dimensions and a singularity spectrum. One can say that self-similarity of multifractals is scale dependent resulting in the singularity spectrum. A multifractal is therefore in a certain sense like a set of intertwined fractals.

2.1 Multifractal Characteristics

The comparison of the main characteristics of fractals and multifractals are summarized below:

Fractal	Multifractal
A measure (volume) V of a set as a function of size l	A (probability) measure versus singularity strength, α
$V(l) \sim l^{D_\mathrm{F}}$	$p_i(l) \propto l^{\alpha_i}$
The number of elements of size l needed to cover the set	The number of elements in a small range from α to $\alpha + \mathrm{d}\alpha$
$N(l) \sim l^{-D_\mathrm{F}}$	$N_l(\alpha) \sim l^{-f(\alpha)}$
The fractal dimension	The multifractal singularity spectrum
$D_\mathrm{F} = \lim\limits_{l\to 0} \frac{\ln N(l)}{\ln 1/l}$	$f(\alpha) = \lim\limits_{\varepsilon\to 0} \lim\limits_{l\to 0} \frac{\ln[N_l(\alpha+\varepsilon) - N_l(\alpha-\varepsilon)]}{\ln 1/l}$

The generalized dimension

$$D_q = \frac{1}{q-1} \lim_{l\to 0} \frac{\ln \sum_{i=1}^{N} (p_i)^q}{\ln l}$$

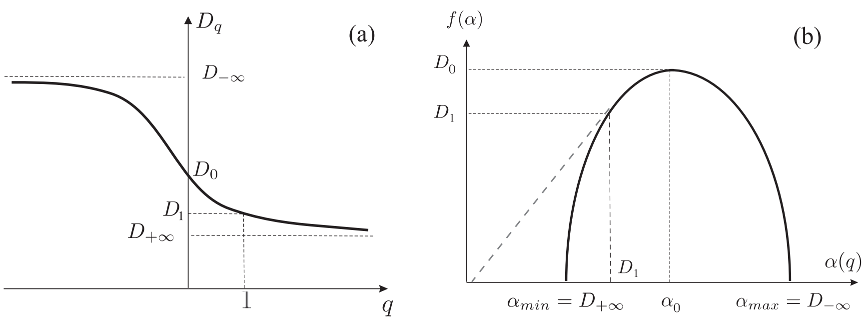

Fig. 1. (a) The generalized dimensions D_q as a function of any real q, $-\infty < q < +\infty$, and (b) the singularity multifractal spectrum $f(\alpha)$ versus the singularity strength α with some general properties: (1) the maximum value of $f(\alpha)$ is D_0; (2) $f(D_1) = D_1$; and (3) the line joining the origin to the point on the $f(\alpha)$ curve, where $\alpha = D_1$ is tangent to the curve.

The generalized dimensions D_q as a function of index q ($D_0 = D_\mathrm{F}$) provide information about multifractality of a given system [7]. Alternatively, we can describe intermittent turbulence by using the singularity spectrum $f(\alpha)$ as a function of a singularity strength α, which also quantifies multifractality of a given system. These functions sketched in Figure 1 describe singularities occurring in considered probability measure, allowing a more clear theoretical interpretation by comparing the experimental results with those obtained from phenomenological models of turbulence [8].

We now define a one-parameter q family of (normalized) generalized pseudo-probability measures [9, 10]

$$\mu_i(q,l) \equiv \frac{p_i^q(l)}{\sum_{i=1}^N p_i^q(l)} \tag{1}$$

With an associated fractal dimension index $f_i(q,l) \equiv \log \mu_i(q,l)/\log l$ for a given q the multifractal singularity spectrum of dimensions is defined directly as the average taken with respect to the measure $\mu_i(q,l)$ in Eq. (1) denoted by $\langle \ldots \rangle$

$$f(q) \ \equiv \ \lim_{l\to 0} \sum_{i=1}^{N} \mu_i(q,l) \ f_i(q,l) = \lim_{l\to 0} \frac{\langle \log \mu_i(q,l) \rangle}{\log(l)} \tag{2}$$

and the corresponding average value of the singularity strength

$$\alpha(q) \ \equiv \ \lim_{l\to 0} \sum_{i=1}^{N} \mu_i(q,l) \ \alpha_i(l) = \lim_{l\to 0} \frac{\langle \log p_i(l) \rangle}{\log(l)}. \tag{3}$$

3 Multifractal Model

Let us consider the generalized weighted Cantor set [6]. Here this set with weight p and two scales is schematically shown in Figure 2 (see Figure 2 of Ref. [1]). This simple example of multifractals provides a useful mathematical language for complexity of turbulent dynamics [3]. Namely, at each stage of construction of this generalized Cantor set we have two rescaling parameters l_1 and l_2, where $l_1 + l_2 \le L = 1$ (normalized) and two different probability measure $p_1 = p$ and $p_2 = 1 - p$.

The difference of the maximum and minimum dimension, associated with the least dense and most dense regions in the considered probability measure, is given by [1, 3]

$$\Delta \equiv \alpha_{\max} - \alpha_{\min} = D_{-\infty} - D_{\infty} = \left| \frac{\log(1-p)}{\log l_2} - \frac{\log(p)}{\log l_1} \right|. \tag{4}$$

The degree of multifractality Δ is naturally related to the deviation from a strict self-similarity. Thus Δ is also a measure of intermittency, which is in contrast to self-similarity, see chapter 8 of Ref. [8]. Moreover, using the value of the strength of singularity α_0 at which the singularity spectrum has its maximum $f(\alpha_0) = 1$ we define a measure of asymmetry by

$$A \equiv \frac{\alpha_0 - \alpha_{\min}}{\alpha_{\max} - \alpha_0}. \tag{5}$$

Please note that the value $A = 1$ ($l_1 = l_2 = 0.5$) corresponds to the one-scale symmetric case, e.g., for the so-called p-model of Ref. [11].

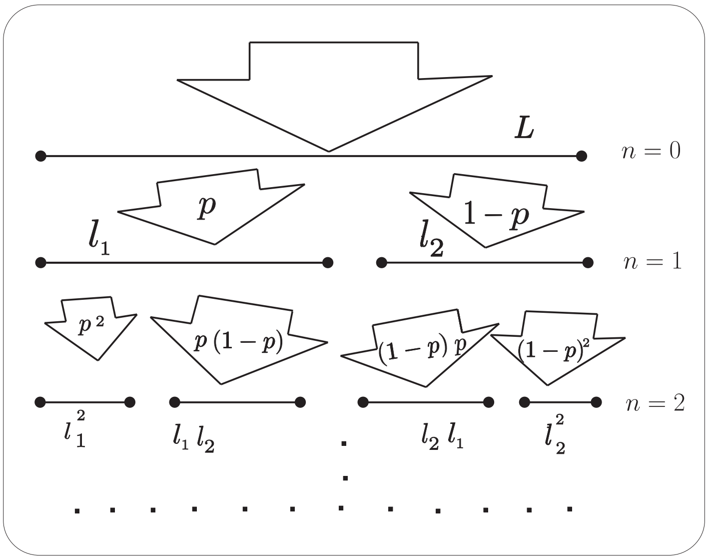

Fig. 2. Generalized two-scale weighted Cantor set model for turbulence.

4 Voyager Data

The trajectories of Voyager 1 and 2 spacecraft are shown in Figure 3, where crossings of the termination shock (TS) and the heliopause (HP) by Voyager 1 are also marked; Voyager 2 is still in the heliosheath and is expected to cross the heliopause in several years. Here we analyse time series of the magnetic field fluctuations measured by Voyager 1 at a wide range of distances before the termination shock crossing (during years 1980–2003), namely between 7 and 90 AU from the Sun, and subsequently (2005–2010) at 95–115 AU, i.e., in the heliosheath, ahead of the heliopuase. This is an extension of our previous analysis [4], ahead of the heliopause.

The generalized multifractal measures can now be constructed using magnetic field strength fluctuations in the following way [12]. Namely, given the normalized time series $B(t_i)$, where $i = 1, \ldots, N = 2^n$ (we take $n = 8$), to each interval of temporal scale Δt (using $\Delta t = 2^k$, with $k = 0, 1, \ldots, n$) we associate some probability measure

$$p(x_j, l) \equiv \frac{1}{N} \sum_{i=1+(j-1)\Delta t}^{j\Delta t} B(t_i) = p_j(l), \tag{6}$$

where $j = 2^{n-k}$, i.e., calculated by using the successive (daily) average values $\langle B(t_i, \Delta t)\rangle$ of $B(t_i)$ between t_i and $t_i + \Delta t$. At a position $x = v_{\mathrm{sw}} t$, at time t, where v_{sw} is the average solar wind speed, this quantity can be interpreted as

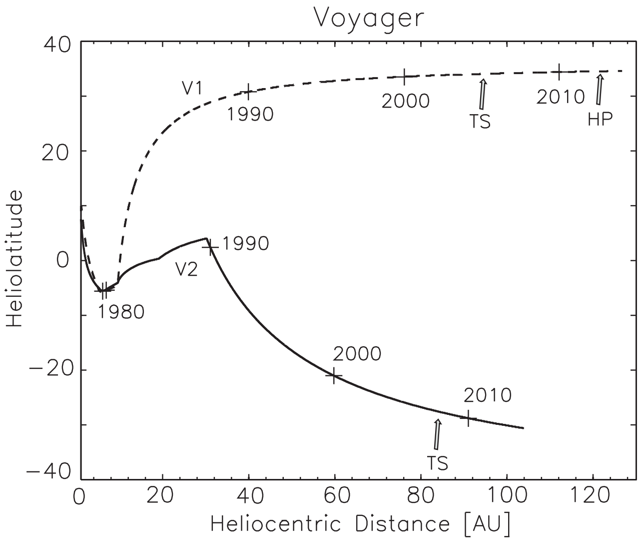

Fig. 3. The heliospheric distances from the Sun and the heliographic latitudes during each year of the Voyager mission. Voyager 1 and 2 spacecraft are located above and below the solar equatorial plane, respectively.

a probability that the magnetic flux is transferred to a segment of a spatial scale $l = v_{\rm sw}\Delta t$ (Taylor's hypothesis).

In the inertial range the average value of the qth moment of the magnetic field strength B at various scales $l = v_{\rm sw}\Delta t$ should scale as

$$\langle B^q(l) \rangle \sim l^{\gamma(q)}, \tag{7}$$

with the exponent $\gamma(q) = (q-1)(D_q - 1)$, see Ref. [12].

5 Results

Following Burlaga [12] for a given q, using the slopes $\gamma(q)$ of $\log_{10}\langle B^q \rangle$ versus $\log_{10} l$ in the range of time scales from 2 to 16 days, one can obtain the values of D_q as a function of q according to Equation (7). Equivalently, as discussed in Sec. 2, the multifractal spectrum $f(\alpha)$ as a function of scaling indices α indicates universal multifractal scaling behaviour, see [3–5]. In this paper the results for the multifractal spectrum $f(\alpha)$ obtained using the Voyager 1 data of the solar wind magnetic fields in the heliosheath at various distances before crossing the heliopause, (a) 94–97 AU, (b) 105–107 AU, (c) 108–111 AU, and (d) 112–115 AU are presented in Figures 4 (a), (b), (c), and (d), correspondingly, cf. [4, 5]. The values are calculated for the weighted two-scale (continuous lines) model

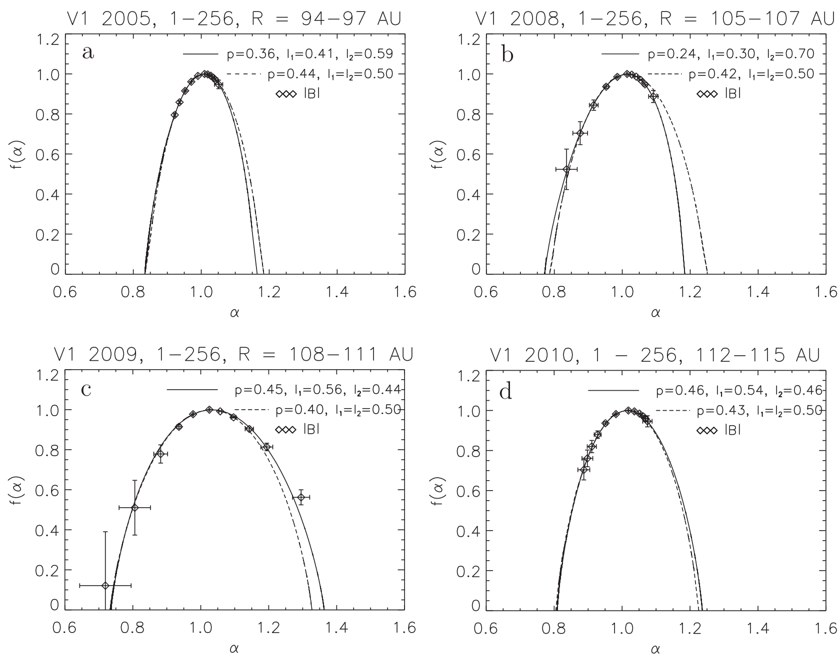

Fig. 4. The singularity spectrum $f(\alpha)$ as a function of a singularity strength α. The values are calculated for the weighted two-scale (continuous lines) model and the usual one-scale (dashed lines) p-model with the parameters fitted using the magnetic fields (diamonds) measured by Voyager 1 in the heliosheath at various distances before crossing the heliopause, (a) 94–97 AU, (b) 105–107 AU, (c) 108–111 AU, and (d) 112–115 AU, correspondingly.

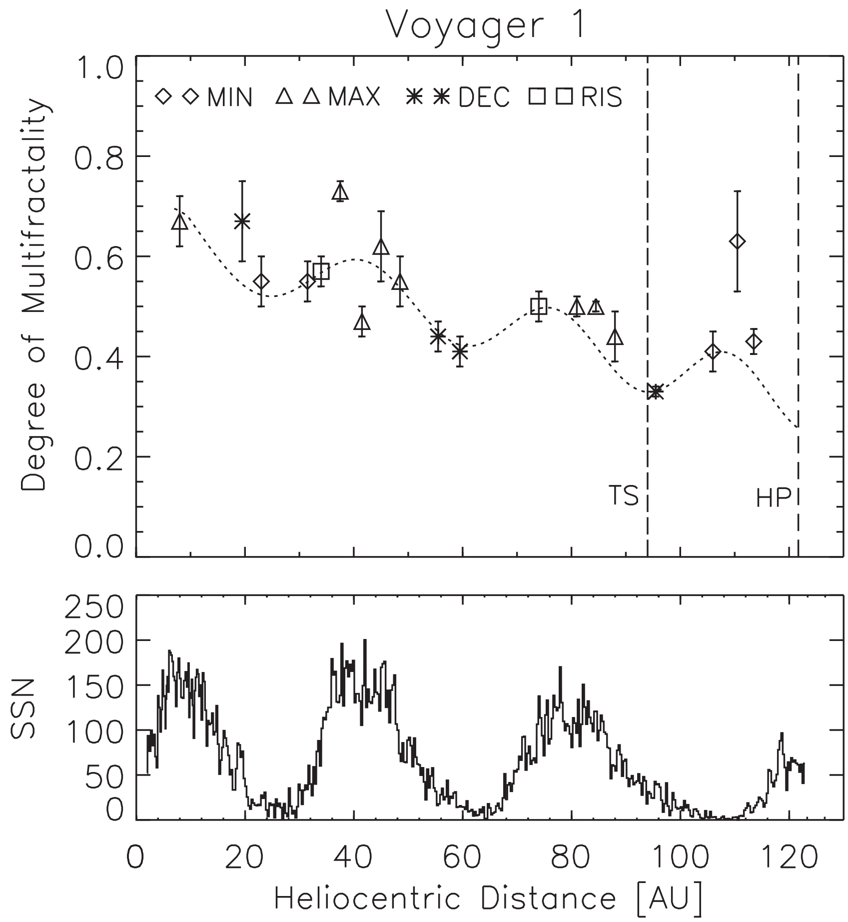

Fig. 5. The degree of multifractality Δ in the heliosphere versus the heliospheric distances compared to a periodically decreasing function (dotted) during solar minimum (MIN) and solar maximum (MAX), declining (DEC) and rising (RIS) phases of solar cycles, with the corresponding averages shown by continuous lines. The crossing of the termination shock (TS) and the heliopause (HP) by Voyager 1 are marked by vertical dashed lines. Below is shown the Sunspot Number (SSN) during years 1980–2010.

and the usual one-scale (dashed lines) p-model, as discussed in Sect. 3, with the parameters fitted using the magnetic fields (diamonds).

We are also looking for the degree of multifractality Δ in the heliosphere as a function of the heliospheric distances during solar minimum (MIN), solar maximum (MAX), declining (DEC) and rising (RIS) phases of solar cycles. The obtained values of Δ roughly follow the fitted periodically decreasing function of time (in years, dotted) with the corresponding averages shown in Figure 5 by continuous lines, cf. [4]. The crossings of the termination shock (TS) and the heliopause (HP) by Voyager 1 are marked by vertical dashed lines. Below are shown the Sunspot Numbers (SSN) during years 1980–2010. We see that the degree of multifractality falls steadily with distance (possibly toward monofractal behaviour, $\Delta = 0$) and is apparently modulated by the solar activity. We see now that further in the heliosheath the degree of multifractality basically follows the periodic dependance fitted inside the heliosphere except a somewhat higher value of Δ obtained now by us near 110 AU, Figure 4 (c).

We have demonstrated that the multifractal scaling is asymmetric in the outer heliosphere [3]. Now, we see that the degree of asymmetry A of this multifractal spectrum in the heliosphere as a function of the phases of solar cycles is shown in Figure 6; the value $A = 1$ (dotted) corresponds to the one-scale symmetric model. Since the density of the measure $\varepsilon \propto l^{\alpha-1}$, this corresponds to its critical value where $\alpha = 1$. One sees that in the heliosphere only one of three points above unity is at large distances from the Sun, cf. [4]. In fact, inside the outer heliosphere prevalently $A < 1$ and only once (during the declining phase) the left-skewed spectrum ($A > 1$, $\alpha < 1$) was clearly observed. Anyway, it seems that the right-skewed spectrum ($A < 1$, $\alpha > 1$) before the crossing of the termination shock is preferred. As expected the multifractal scaling is asymmetric before shock crossing with the calculated degree of asymmetry at distances $70 - 90$ AU equal to $A = 0.47 - 0.96$. It also seems that the asymmetry is probably changing when crossing the termination shock ($A = 1.0 - 1.5$) as is also illustrated in Figure 6, which is in fact related to changing properties of the magnetic field density ε at the termination heliospheric shock. However, in spite of large errors bars and a very limited sample, it seems that symmetric spectrum is preferred in the heliosheath [4], except of a possibly asymmetric spectrum near 110 AU, Figure 4 (c).

For comparison, the values calculated from the papers by Burlaga et al. [13] and [14] (LB) are also given in Table 1, cf. [4]. One sees that the degree of multifractality for fluctuations of the interplanetary magnetic field strength obtained from independent types of studies are in good agreement, except somewhat higher value of Δ near 110 AU, cf. Fig. 7 of Ref. [14]. Generally these values are smaller than that for the energy rate transfer in the turbulence cascade ($\Delta = 2 - 3$), see Ref. [3].

Moreover, it is worth noting that our values obtained before the shock crossing, $\Delta = 0.4 - 0.7$, are somewhat greater than those after shock crossing $\Delta =$ 0.3–0.4. In this way we have provided a supporting evidence that the magnetic field behaviour in the outer heliosphere, even in a very deep heliosphere, may

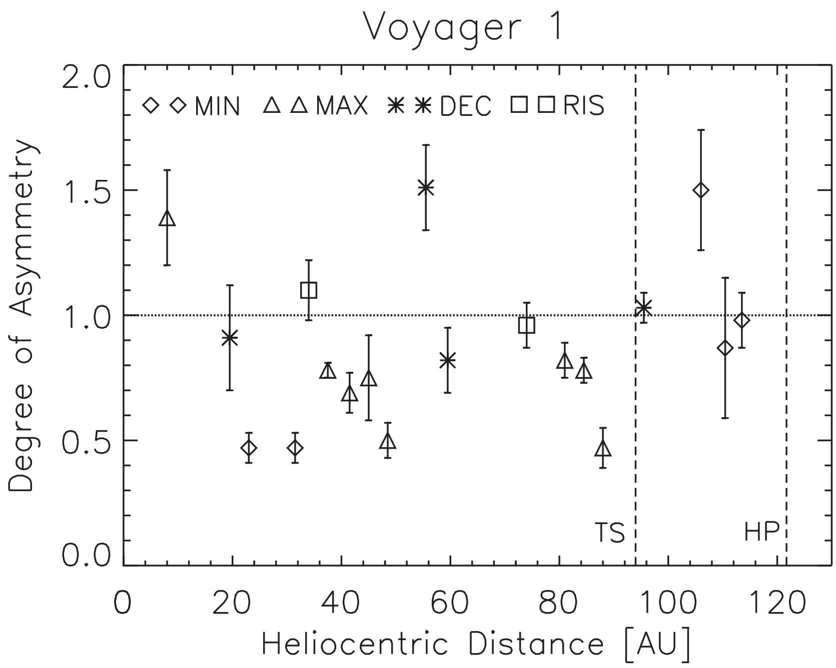

Fig. 6. The degree of asymmetry A of the multifractal spectrum in the heliosphere as a function of the heliospheric distance during solar minimum (MIN) and solar maximum (MAX), declining (DEC) and rising (RIS) phases of solar cycles with the corresponding averages denoted by continuous lines; the value $A = 1$ (dotted) corresponds to the one-scale symmetric model. The crossing of the termination shock (TS) and the heliopause (HP) by Voyager 1 are marked by vertical dashed lines.

Table 1. The degree of multifractality Δ and asymmetry A of the multifractal spectrum for the magnetic field strength observed by Voyager 1 at various heliospheric distances, before and after crossing the termination shock, as calculated by L. Burlaga (LB) and by the author of this paper (WM).

Heliocentric Distance	**Year**	**Multifractality** Δ (WM)	Δ (LB)	**Asymmetry** A(WM)	A (LB)
$7-40$ AU	1980-1989	$0.55-0.73$	0.64	$0.47-1.39$	0.69
$40-60$ AU	1990-1995	$0.41-0.62$	0.69	$0.51-1.51$	0.63
$70-90$ AU	1999-2003	$0.44-0.50$	0.69	$0.47-0.96$	0.63
$95-107$ AU	2005-2008	$0.33-0.41$	0.34	$1.03-1.51$	0.89
$108-115$ AU	2009-2010	$0.44-0.63$	0.34	$0.87-0.98$	1.0

exhibit a multifractal scaling, while in the heliosheath smaller values indicate possibility toward a monofractal behaviour, implying roughly the same density of the probability measure. Time series analysis of Voyager 1 data in the very local interstellar medium will be a subject of our further studies [15].

6 Conclusions

Using our weighted two-scale Cantor set model, which is a convenient tool to investigate the asymmetry of the multifractal spectrum, we confirm the characteristic shape of the universal multifractal singularity spectrum. In fact, as seen in Figure 4, $f(\alpha)$ is a downward concave function of scaling indices α.

We show that the degree of multifractality for magnetic field fluctuations of the solar wind falls steadily with the distance from the Sun and seems to be modulated by the solar activity.

Moreover, we have considered the multifractal spectra of fluctuations of the interplanetary magnetic field strength before and after shock crossing by Voyager 1, at 84 and 94 AU from the Sun, correspondingly. In contrast to the right-skewed asymmetric spectrum with singularity strength $\alpha > 1$ inside the heliosphere, the spectrum becomes more left-skewed, $\alpha < 1$, or roughly symmetric after the shock crossing in the heliosheath, where the plasma is expected to be roughly in equilibrium in the transition to the interstellar medium.

We also confirm that before the shock crossing, especially during solar maximum, turbulence is more multifractal than that in the heliosheath. In addition, outside the heliosphere during solar minimum the spectrum seems to be dominated by values of $\alpha \sim 1$. This is very interesting because it represents a first direct information of interest in the astrophysical context beyond the heliosphere. We can thus expect that our results, obtained *in situ* rather than through scintillations observations, are relevant in the context of interstellar turbulence.

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