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APPLICATION OF THE RADIUS-VECTOR OF ARTIFICIAL SATELLITE AS LENGTH MEASURE FOR GEODETIC PURPOSES

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S u m m a r y

By utilizing the third Kepler's Law, it is possible to define the semi-axis of satellite orbit on the basis of the observed revolution period. If the eccentricity of the orbit is also known, then - assuming it to be near zero - it is possible to compute the radius-vector of the satellite for an arbitrary moment, even if the remaining elements were known only approximately.

The present study is devoted, first of all, to the examination of the degree of accuracy suspectible to be achieved in the determination of the radius-vector. Assuming that the orbital eccentricity is near zero, that the altitude of perigee is 1000-3000 km and that the satellite has a small area/mass ratio, the radius-vector can be computed as shown in Chapter III, with an accuracy of 10^{-5}. A poor knowledge of coefficients of tesseral harmonics of Earth's gravity field seems to be essential impediment in achieving a higher precision. The other sources of perturbations, such as: coefficient errors of zonal harmonics, atmospheric drag, solar radiation pressure, solar and lunar attraction may be regarded as lesser disturbance causes. Also the way of observation and computation of orbital elements, eliminating the effect of errors of the observing site coordinates is showed.

The Chapter V presents some modes of using the known radius-vector for determining the coordinate points on the Earth's surface in the geocentric system oriented in conformity with the direction of the revolution axis and the equatorial plane.

C h a p t e r I

C o n c e p t i o n o f t h e m e t h o d a n d i t s m a i n d e p e n d e n c e s

We know from the experience of natural sciences that the time intervals may be measured with a considerably higher accuracy than the distances. Clocks that are able to maintain during several weeks their rate constancy of the order of 10^{-9}, do not belong any more to exceptions today, while the geodetic measurements of an accuracy of 10^{-6} represent, again and again, a complicated technical problem. That is why, numerous attempts are being made with a view to replacing the direct method of distance measurement by time measurement methods, which is, in a given physical phenomenon, the function of length of the segment involved. Among these methods range the measurements of distances made with the help of radio and light waves propagating with the known speed.

The astronomy knows another phenomenon presenting a strictly determined relation between the time and the distance - this is the undisturbed Kepplerian motion in which the semi-major axis of the orbit a is related to the period of revolution P by the third Law of Kepler:

$$\frac{a^3}{P^2} = \text{const} \qquad /1.1/$$

The leading idea of the present study is to show the attempts of utilizing this dependence. Of course, artificial satellites will be called into play here, for their orbits have dimensions comparable to those of the Earth and may be shaped by the man in order to attain optimal conditions. In addition, the artificial satellites first of all remain under the influence of the gravitation field of the Earth, this allowing for the other essential feature of this method to be stated, i.e. the possibility of referring the involved measurements to the center of the Earth's mass.

One of the main tasks of the contemporary higher geodesy is to obtain possibilities of determining homogeneous coordinates of points on the whole Earth's surface in a system whose position with regard to the fundamental physical elements of the Earth body - its mass center and axis of revolution - would be known. A particular and the most advantageous case would be offered by a system whose origin will coincide with the center of the Earth's mass, and one of its axis - with the Earth's revolution axis. An additional condition should be that the scale of such a World geodetic system be the same everywhere.

The fulfillment of the above conditions by means of methods of the classical geodesy proves to be difficult not only because of the distance between the different continents or the separate geodetic systems or the necessity of gathering an immense quantity of observational data, but also because the computation and adjustment of extensive triangulation networks require the adoption of one or another hypothesis concerning the inner structure of the Earth. The appearance of satellite methods have opened new and vast possibilities. Using the satellite geometric methods such as satellite triangulation, we perform the measurements of directions in the system of astronomical coordinates - the declination and the right ascension - it means, in such a system whose orientation with regard to the axis of the Earth's revolution, the instantaneous or the mean one, is exactly known. In the dynamic methods, the center of the Earth's mass is directly involved. A review and the discussion of results obtained by this method are presented in paper of Zieliński [1967]. The work done so far does not, of course, exhaust all possibilities offered by satellite methods. It is, therefore, desirable to proceed to the improvement of the existing and the search for new methods in this domain.

The method proposed in this study is different from methods applied so far. In those methods, the quantities serving as starting points for further computations and ultimate results were always the instantaneous three-dimensional coordinates of the satellite. Comparison, adjustment and transformation of these coordinates led to the final result represented by the geocentric coordinates of the observing station. In our method, the only basic element will be the radius-vector. Its length changes much more slowly - especially in orbits of small eccentricity - than the coordinates which vary literally with a cosmic velocity. The idea of this method has sprung up in 1963, after Poland had taken the engagement to count the ephemerides of the satellite Alouette /1962 Beta Alfa 1/ for the socialist countries participating in the multilateral scientific cooperation. The project for the respective observations, described in Chapter V, was submitted to the Conference of Observers of Artificial Earth Satellites in Moscow, 1963. However, the specificity of this kind of research work is, that it cannot be performed on a small scale; if we want to obtain reliable scientific results, the collaboration of a great number of observing and computing agencies is required, this being possible only within the framework of an international cooperation. Before undertaking and, all the more, before initiating such a collaboration, a thorough analysis of the whole problem is indispensable. During the course of the work, this problem has evolved to the present - a more general - form.

In this paper the application of the theory of radius-vector is being treated in broad outlines. Among the projects mentioned in Chapter V only one has been tested by using the substitutional observational data: the synchronous observations of Echo I. The main topic is being devoted to the theory itself and to the problem: which perturbations ought to be taken into account and what is the accuracy with which we are able to compute them?

Let us now go over to a brief discussion of the theory.

An element characterzing the linear dimensions of the orbit is its semi-major axis a. We shall determine it by measuring the period of revolution or - what comes to the same - the mean motion n. Introducing the mean motion, we can write down the third Kepler's Law:

$$n^2 \cdot a^3 = \mu \qquad /1.2/$$

where: $n = \frac{2\pi}{P}$.

In this formula μ designates the mass of the Earth /the case being confined to the motion of artificial satellites/ multiplied by the gravitation constant. Let us first see what effect the error of the period measurement will have on the accuracy of a.

Differentiating /1.2/ we have

$$d_a = \frac{\mu P \, dP}{6 a^2 \pi^2} \qquad /1.3/$$

Assuming that $dP = 0^s.01$, this being not at all an exorbitant accuracy for observing possibilities, we obtain for orbits at altitudes from 1000 to 3000 km:

$$d\,a = 7 \text{ m.}$$

It follows from the ulterior analysis as well as from numerous publications /for instance: Satellite Orbital Data SAO Spec. Rept./, that the accuracy of period equal to $0^s.01$, this corresponding to the accuracy of $n \cdot 10^{-6}$, would be rather the inferior limit of accuracy. Whereas, the error: 7 m corresponding also to the relative accuracy: 10^{-6}, is from the geodetic point of view quite satisfactory. A further reduction of the error of $d\,a$ will give no effects because it would be below the level of observation errors.

The quantity μ is also encumbered with a certain error. The latter has the following bearing upon the semi-axis:

$$d\,a = \frac{a}{3} \frac{d\mu}{\mu} \qquad /1.4/$$

Assuming that μ is equal to 398603 $km^3 sec^{-2}$ and that $d\,\mu$ is equal to 3 $km^3 sec^{-2}$, we obtain for

H = 1000 km, da = 18 m
H = 3000 km, da = 23 m.

The quantity μ and its accuracy is of a particular importance for the present theory. It defines the scale of the system or of the geometrical construction which will be considered in some concrete case. Anyway, the error of μ will appear as a systematic error not bringing about local deformations of a given construction. We derive the radius-vector from the semi-major axis using the formula:

$$r = a\,(1 - e \cdot \cos E) \qquad /1.5/$$

This brief discussion leads to the conclusion that the accuracy we have a chance of attaining by this method is of the order of 10^{-6}, this amounting to 7 m - 10 m, according to the altitude of orbit. And so, we shall adopt the following rule with regard to the need of introducing perturbation corrections and their accuracy: corrections will be considered negligible, if they give a perturbation smaller than one meter in radius-vector, while they will be considered sufficiently precise, if the accuracy is better to one meter.

However, not all types of orbits and, naturally, not all types of satellites can be chosen for this method. In order to avoid rapid changes in the length r, we shall utilize orbits with small eccentricities of the type ANNA Ib and ALOUETTE, those being smaller than 0.01. For possible elimination of disturbing effects of the atmosphere, we shall fix the inferior limit of the altitude to 1000 km over the Earth's surface and the upper limit - to 3000 km. Further, we admit that the satellite to be dealt with is of the type called heavy vehicle, of a low area/mass ratio, similar to ANNA or ALOUETTE.

The real motion of an artificial Earth satellite differs, however, quite distinctly from the Keplerian motion - considerably more than motions of planets, for instance. The greatest perturbations are provoked by the very fact that not only the mass of our globe is not concentrated at its center but that it does neither represent a uniform sphere nor even a globe composed of homogeneous spherical layers. In reality, the Earth has rather the form of an ellipsoid and even this should be treated as an approximation. In more accurate approaches the Earth and, consequently, the field of its gravitation potential dependent upon the distribution of the mass should be approximated by series expansions. In this connection the Kepler orbit can serve us merely as an auxiliary concept not actualized in reality. A Keplerian orbit which is osculatory to the real trajectory of the satellite at a given moment t, i.e. an orbit such as would develop if at that moment the diturbing forces suddenly disappeared, will be called o s c u l a t i n g o r b i t. So, we can imagine that the real orbit of a satellite consists of infinitesimal segments of osculating orbits, or that it is similar to the Keplerian orbit, yet with varying elements. Changes of these elements will be termed perturbations of elements.

Among the perturbations, we shall distinguish the followings ones: s e c u l a r perturbations progressively varying with time and p e r i o d i c perturbations. The latter are divided into: short-period perturbations occurring during one period of revolution, diurnal perturbations appearing during the day of the orbital plane /the period of time during which the Earth turns with regard to the orbital plane about 360 degrees/, and l o n g - p e r i o d perturbations with a period equal to the period of change in the perigee argument about 2π .

In our practice with satellites, we have to do - next to the concept of the osculating orbit - with the mean elements. Different authors /Zhongolovich and Pellinen [1962] , Tchebotariev [1963] , Gaposchkin [1964] / are using this designation for different quantities, hence a precise definition appears to be necessary. We are going to adopt the definition given by Zhongolovich and used also by Gaposchkin, which seems to be the most convenient in our case. The p e r t u r b a t i o n c o r r e c t i o n will mean a correction including only short-period and diurnal perturbations. The mean element will be equal to the osculating element less the perturbation correction, i.e.

$$\mathcal{E}_o = \mathcal{E}_m + \delta_{\mathcal{E}} \qquad /1.6/$$

where $\mathcal{E}_o$ - osculating element
$\mathcal{E}_m$ - mean element
$\delta_{\mathcal{E}}$ - perturbation correction.

That way, the changes of mean elements are cumulating secular variations produced both by the gravitation field of the Earth and any other possible cause.

Also the term "period of revolution" has various meanings. Zhongolovich [1960] defines the anomalistic period as an interval of time between two subsequent passages of the satellite through the perigee. Kozai [1959] uses an expression for the mean motion, defining by the same period:

$$a = \left\{\mu . a^{-3} \left[1 - \frac{1}{3} \frac{C_{20}}{p^2} \left(1 - \frac{3}{2} \sin^2 i\right) . \sqrt{1-e^2}\right]\right\}^{1/2} \qquad /1.7/$$

where: a - is semi-axis of mean orbit, p - orbit parameter; thus, Kozai is including into it also the secular variation of the mean anomaly in epoch. For our purposes, the anomalistic period will signify the period of the osculating orbit in a given epoch, connected with the semi-major axis of that orbit by the equation:

$$n^2 a^3 = \mu$$

For computing the length of the radius-vector, we are going to adopt the following assumption: we shall admit that we have at our disposal an appropriate set of observations well located in space and time, permitting for the mean elements of the orbit to be computed with satisfactory accuracy, as well as their variations. Thus, we cease dealing with all the long-period perturbations, since they would be contained in variations of mean elements. Only the short-period and diurnal perturbations will need to be taken into account. It seems purposeful to indicate here that we are using for the computation of perturbations caused by consecutive harmonics of the terrestrial potential of gravitation, the values of coefficients of harmonics determined from satellite observations. Yet we do by no means enter in this way the "vicious circle", since those determinations are mainly based on observed secular variations, which are many times greater than short-period variations computed here.

Chapter II

COMPUTATION OF THE ORBIT AND ITS ACCURACY

The modes of computing orbits of artificial satellites from observations of directions - and only such will be here dealt with - are merely slightly modified methods of the classical astronomy , used for the calculation of orbits of minor plane and comets. We shall not discuss here the approximate methods, such as the Gauss´s or Laplace´s methods applied for the determination of approximate elements on the basis of several observations. But let us give more time to the method destined for the most accurate computation of elements from the observational data available, the so-called i m p r o v e m e n t of orbits. The principle of this method consists in expressing corrections to the observed coordinates α and δ by the function of corrections to the orbital elements:

$$\Delta c = \sum \frac{\partial c}{\partial \vartheta_i} \cdot \Delta \vartheta_i : \qquad /2.1/$$

where - Δc is the correction to the coordinate

ϑ_i - the orbital element.

That way is forming the observation equations in which the free terms are the differences between the observed coordinates and the computed from the approximate elements /o - c/, the unknowns - corrections to those approximate elements. Adopting the observed coordinates α and δ and introducing the three-dimensional coordinate system x,y,z, we obtain:

$$\left.\begin{aligned} \Delta\alpha &= \sum \frac{\partial \alpha}{\partial x} \cdot \frac{\partial x}{\partial \vartheta_i} \cdot \Delta\vartheta_i + \sum \frac{\partial \alpha}{\partial y} \cdot \frac{\partial y}{\partial \vartheta_i} \cdot \Delta\vartheta_i \\ \Delta\delta &= \sum \frac{\partial \delta}{\partial x} \cdot \frac{\partial x}{\partial \vartheta_i} \Delta\vartheta_i + \sum \frac{\partial \delta}{\partial y} \cdot \frac{\partial y}{\partial \vartheta_i} \cdot \Delta\vartheta_i + \sum \frac{\partial \delta}{\partial z} \cdot \frac{\partial z}{\partial \vartheta_i} \cdot \Delta\vartheta_i \end{aligned}\right\} \qquad /2.2/$$

It results from the very geometry of the orbit that the accuracy in its determination depends not only on the accuracy and the number of observations but also on their distribution.

A precise determination of the ascending node will not be possible if the observations are concentrated in the proximity of $u = 90^{\circ}$ or 270°; nor shall we be able to determine properly the eccentricity if only a small arc of orbit is being observed. With that, the distribution of observations depends mainly upon the visibility conditions, and so, the altitude of the orbit, its orientation with regard to the direction of the Sun. Yet for our purposes, we shall assume that the observations are uniformly distributed at least on the arc of orbit = 180°. According to the analysis given by Sotchilina [1963] in her paper devoted to the accuracy of the determination of orbits, in such a case all elements will be found with an accuracy of the same order. Sotchilina calculates the coefficients of weight of the particular unknowns Q_{ii} as follows:

$$\left.\begin{aligned} Q\,(M_0) &= \frac{9}{4}\,\frac{1}{Na^2} \\ Q\,(n) &= \frac{75}{4}\,\frac{1}{Na^2t^2} \\ Q\,(\Omega) &= \frac{2}{Na^2} \\ Q\,(i) &= \frac{2}{Na^2} \\ Q\,(ax) &= \frac{1}{2Na^2} \\ Q\,(ay) &= \frac{1}{2Na^2} \end{aligned}\right\} \qquad /2.3/$$

where: N - number of observations, t - observation period expressed in days, $a_x = e \cos \omega$, $a_y = e \sin \omega$.

The accuracy of the respective elements is calculated with the help of formulae:

$$m_{\ni_i} = m_o \cdot \sqrt{Q_{ii}} \qquad /2.4/$$

$$m_o = \sqrt{\frac{\sum^{N}[(\varrho . \cos \delta \Delta\alpha)^2 + (\varrho \Delta \delta)^2]}{2N.6}} \qquad /2.5/$$

where: ϱ is the topocentric distance of the satellite.

Adopting the accuracy of observations as being:

$$m_{\delta} = \cos \delta \quad m_{\alpha} = 2'' ,$$

this corresponding to:

$$\varrho m_{\delta} = \varrho \cos \delta \quad m_{\alpha} \cong a . 10^{-6}$$

we shall have:

$$m_o = a . 10^{-6} \sqrt{\frac{2 N}{2 N-6}} \cong a . 10^{-6}$$

If we take $N = 100$ and $t = 10^d$, we obtain:

$$\left.\begin{aligned} m_{Mo} &= 1,5.10^{-7} \text{ rad} \\ m_n &= 0,5.10^{-7} \text{ rad/d} \\ m_{\Omega} &= 1,3.10^{-7} \text{ rad} \\ m_i &= 1,3.10^{-7} \text{ rad} \\ m_{ax} &= 0,3.10^{-7} \\ m_{ay} &= 0,3.10^{-7} \\ m_e &= m_{ax} = m_{ay} \end{aligned}\right\} \qquad /2.6/$$

since

$$de = da_x \cos \omega + da_y \sin \omega \qquad /2.7/$$

$$m_e^2 = m_{ax}^2 \left(\cos^2 \omega + \sin^2 \omega\right) \qquad /2.8/$$

It results from the above development that it is possible to attain a very high accuracy of the order of 10^{-7} in the determination of orbital elements - provided that the observations are properly distributed and their number and accuracy being as assumed.

Still, we did not have here into account a factor having an essential bearing, it means, the errors of observing station coordinates. This factor does not appear in problems of the classical astronomy, because of the great distances between the observer and the body observed. The question is different with artificial satellites; having to deal here with the geocentric motion, it is necessary to know the geocentric coordinates of observing stations.

In order to evaluate the effect of errors of observing station coordinates, let us start with the basic equations determining the position of the satellite in the Cartesian geocentric system, on the ground of the known topocentric coordinates α and δ :

$$\left.\begin{aligned} x &= X + \varrho . \cos (\alpha - \Omega) . \cos \delta \\ y &= Y + \varrho . \sin (\alpha - \Omega) . \cos \delta \\ z &= Z + \varrho . \sin \delta \end{aligned}\right\} \qquad /2.9/$$

where:

x, y, z = coordinates of satellite,
X, Y, Z = coordinates of observing stations,
α, δ = spherical coordinates observed,
ϱ = topocentric radius-vector,
Ω = right ascension of the ascending node

assuming that the axis x coincides with the line of nodes. Coordinates of the observing point are expressed by formulae:

$$\left.\begin{aligned} X &= R . \cos \varphi' . \cos (s - \Omega) \\ Y &= R . \cos \varphi' . \sin (s - \Omega) \\ Z &= R . \sin \varphi' \end{aligned}\right\} \qquad /2.10/$$

where:

s = sidereal time of the observing station

φ' = geocentric latitude

R = radius-vector of the observing station.

We may now write total differentials of equation /2.9//treating Ω as constant/:

$$\left.\begin{aligned} dx - dX &= -\rho\cos(\alpha-\Omega)\,.\,\sin\delta\, d\delta - \rho\sin(\alpha-\Omega)\,.\,\cos\delta\, d\alpha + \cos(\alpha-\Omega)\,.\,\cos\delta\, d\rho \\ dy - dY &= -\rho\sin(\alpha-\Omega)\,.\,\sin\delta\, d\delta + \rho\cos(\alpha-\Omega)\,.\,\cos\delta\, d\alpha + \sin(\alpha-\Omega)\,.\,\cos\delta\, d\rho \\ dz - dZ &= +\rho\cos\delta\, d\delta + \sin\delta\, d\rho \end{aligned}\right\} \quad /2.11/$$

wherefrom we can find $d\alpha$ and $d\delta$:

$$\left.\begin{aligned} \cos\delta\, d\alpha &= -\frac{\sin(\alpha-\Omega)}{\rho}(dx-dX) + \frac{\cos(\alpha-\Omega)}{\rho}(dy-dY) \\ d\delta &= \frac{-\cos(\alpha-\Omega)\sin\delta}{\rho}(dx-dX) - \frac{\sin(\alpha-\Omega)\,.\,\sin\delta}{\rho}(dy-dY) + \frac{\cos\delta}{\rho}(dz-dZ) \end{aligned}\right\} \quad /2.12/$$

These equations serve in the so-called orbit improvement process as observation equations in which the left-hand member represents a free term, assuming that $dX = dY = dZ = 0$, and that dx, dy and dz are functions of orbital elements. But dX, dY, dZ being in reality $\neq 0$, their effect will burden the free terms of equations /2.12/. Let us rewrite those equations in the following manner:

$$\left.\begin{aligned} \rho\cos\delta\, d\alpha + (\Delta)_\alpha &= -\sin(\alpha-\Omega)\,dx + \cos(\alpha-\Omega)\,dy \\ \rho\, d\delta + (\Delta)_\delta &= -\cos(\alpha-\Omega)\sin\delta\, dx - \sin(\alpha-\Omega)\,.\,\sin\delta\, dy + \cos\delta\, dz \end{aligned}\right\} \quad /2.13/$$

$(\Delta)_\alpha$ and $(\Delta)_\delta$ will be the effect of errors of observing station coordinates on the particular equations, taking the form of:

$$\begin{aligned} (\Delta)_\alpha &= -\sin(\alpha-\Omega)\ dX + \cos(\alpha-\Omega)\ dY \\ (\Delta)_\delta &= -\cos(\alpha-\Omega)\ \sin\delta\ \ dX - \sin(\alpha-\Omega)\sin\delta\ dY + \cos\delta\ dZ \end{aligned} \quad /2.14/$$

If the coordinates of stations are determined by astronomical methods, we may expect the error to reach $\pm$ 500 m, owing the local deflections of the vertical. Admitting $(\alpha-\Omega) = 315^0$ for the first of equations /2.14/, and 225^0 - for the second one, $\delta = 45^0$, and ρ = 1200 km / of an orbit of the type of Alouette, Anna, Echo I/, we obtain:

$$\left.\begin{aligned} \frac{(\Delta)_\alpha}{\rho\cos\delta} &= \frac{2.0,71.500}{1200000.0,71}\,.\,3438 \cong 2',6 \\ \frac{(\Delta)_\delta}{\rho} &= \frac{0,71.(0,71+0,71+1)\,.\,500}{1200000}\,.\,3438' = 2',4 \end{aligned}\right\} \quad /2.15/$$

So, we can see that - when unfavorable circumstances occur - this effect may be very important, exceeding many times the observation accuracy. It will be, to a certain degree, attenuated by the fact that the coefficients: $\sin(\alpha-\Omega)$, $\cos(\alpha-\Omega)\sin\delta$, $\cos\delta$ will take their values in a way close to an accidental one; what more, dX, dY are not constants, they being dependent upon the angle s /vide formula /2.10//.

From the theoretical point of view the expressions $(\Delta)_\alpha$ and $(\Delta)_\delta$ do not concentrate the total effect of coordinate errors of observing stations, for both ρ and the coefficients including $(\alpha-\Omega)$ and δ are functions of these coordinates

$$\text{ex.g.:}\quad \rho = \sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2} \quad /2.16/$$

This remainder will however be insignificant, and may be neglected in practice. Let us assume that the ρ will be found with an error of $\pm$ 600 m, this giving - when ρ = 1200 km - a relative accuracy of 1/2000. It is with this accuracy that the free terms will be defined and the unknowns determined. Yet, the unknowns dx, dy, dz being of the order of hundreds of meters their determination error will be within the range of one meter this will be, in the same scale, carried over to corrections of elements.

The same situation is with reference to the trigonometric coefficients.

So, there is still the problem, how to proceed in order to eliminate or, at least, to reduce this unfavorable influence. One of the measures could be the utilization of observations of the greatest possible number of stations, so that the errors of station positions take an accidental character. Not always will this, however, be realizable. In such a case, another course of action ought to be applied.

We may anticipate that it will be impossible to eliminate the effect under considera-tion from certain elements - either by computations or by use of special observing methods. Such are the elements which define the orientation of the orbit: the right ascension of the ascending node, the inclination and the argument of the perigee. Those elements depend directly on the coordinate system chosen. The remaining three elements do not depend on the choice of the system, and it may therefore be presumed that we might succeed in eliminating the error effect of station positions wholly or, at least, partially. We shall now show that this is possible for the mean motion and, consequently, for the semi-major axis of the orbit. Let us assume that we have a set of observations made in such a way that at each station the satellite had been observed several times during different passes. We shall consider two of such obser-vations carried out at the same station. Let us set for each of them the equations /2.13/, subtracting them by member accordingly. Then we shall have:

$$\left.\begin{aligned}
&\rho_2 \cos\delta_2\, d\alpha_2 + (\Delta)\alpha_2 - \rho_1 \cos\delta_1\, d\alpha_1 - (\Delta)\alpha_1 = -\sin(\alpha_2 - \Omega_2)\, dx_2 + \\
&\qquad + \cos(\alpha_2 - \Omega_2)\, dy_2 + \sin(\alpha_1 - \Omega_1)\, dx_1 - \cos(\alpha_1 - \Omega_1)\, dy_1 \\
&\rho_2\, d\delta_2 + (\Delta)\delta_2 - \rho_1\, d\delta_1 - (\Delta)\delta_1 = -\cos(\alpha_2 - \Omega_2) \cdot \sin\delta_2\, dx_2 - \sin(\alpha_2 - \Omega_2)\sin\delta_2\, dy_2 + \\
&\qquad + \cos\delta_2\, dz_2 + \cos(\alpha_1 - \Omega_1)\sin\delta_1\, dx_1 + \sin(\alpha_1 - \Omega_1)\sin\delta_1\, dy_1 - \cos\delta_1\, dz_1
\end{aligned}\right\} \quad /2.17/$$

Taking into account the formulae /2.14/, we may write:

$$\left.\begin{aligned}
&(\Delta)\alpha_2 - (\Delta)\alpha_1 = -\sin(\alpha_2 - \Omega_2)\, dX_2 + \cos(\alpha_2 - \Omega_2)\, dY_2 + \sin(\alpha_1 - \Omega_1)\, dX_1 - \cos(\alpha_1 - \Omega_1)\, dY_1 \\
&(\Delta)\delta_2 - (\Delta)\delta_1 = -\cos(\alpha_2 - \Omega_2)\sin\delta_2\, dX_2 - \sin(\alpha_2 - \Omega_2)\sin\delta_2\, dY_2 + \\
&\qquad + \cos\delta_2\, dZ_2 + \cos(\alpha_1 - \Omega_1)\sin\delta_1 + \sin(\alpha_1 - \Omega_1)\sin\delta_1\, dY_1 - \cos\delta_1\, dZ_1
\end{aligned}\right\} \quad /2.18/$$

The effect of errors of observing station coordinates will be eliminated when the above expressions = 0. Let us first consider the second equation /0.18/, as being more simple. Here an adequate condition will be provided by $\delta_1 = \delta_2 = 0$. Since $dZ_1 = dZ_2$, the right-hand member becomes = 0. This is also a necessary condition because, if $\delta_1 \neq \delta_2$, then the coefficient of dZ is $\neq 0$, and if $\delta_1 = \delta_2 \neq 0$, then the coefficients of dX and dY will always be $\neq 0$.

Thus, we may deduce from this that for an utilization of the equation /2.17.2/ as an observation equation for the improvement of the orbit, it is necessary to have observations carried out during the satellite pass through the topocentric equator of the observing sta-tion. Let us assume that this condition has been strictly fulfilled; then the equation /2.17.2/ will take the following form:

$$\rho_2\, d\delta_2 - \rho_1\, d\delta_1 = dz_2 - dz_1 \qquad /2.19/$$

Substituting for dz the formulae connecting them with the corrections of elements in the form given by Sotchilina [1963] we shall have:

$$\begin{aligned}
dz_2 - dz_1 &= (y_2 - y_1)\, di_o + (A_{z2} - A_{z1})\, dU_o + (B_{z2} - B_{z1})\, dax + \\
&+ (C_{z2} - C_{z1})\, day + (A_{z2} \cdot t_z - A_{z1} t_1)\, d\, n_o + (A'_{z2} - A'_{z1})\, dn'_o + (A''_{z2} - A''_{z1})\, dn''_o
\end{aligned} \qquad /2.20/$$

In this equation all pairs of parameters designated with indices $_1$ and $_2$ will consist of elements differring slightly from each other, except for t_1 and t_2. Especially for the Keplerian orbit it will be as follows:

$$\rho_2\, d\delta_2 - \rho_1\, d\delta_1 = A_z\, (t_2 - t_1)\, dn \qquad /2.21/$$

It is clearly seen from the equation /2.21/ that this method helps to determine accurately only the mean motion and its derivatives from precisely measured observation moments. That is why, the following course of action seems to be advisable:

1. Accomplishing the computations according to the normal procedure, applying the formulae given by Sotchilina and utilising the whole set of observations;
2. Computation - with the help of the formula /2.19/ - of corrections $d\bar{n}_0$, dn_0', dn_0'', assuming di_0, dU_0, da_x, da_y to be = 0, and utilizing only such observations which are adapted to the present method;
3. Return to the above formulae and observations, handling now $\bar{n}_0$, n_0', n_0'' as constants, and correcting i_0, U_0, a_x, a_y;
4. Repeted use of the formula /2.19/ a.s.o. until an adequate convergence is attained.

The application of the above method, that is, the use of the equation /2.17.2/ is possible only with satellites with great orbital inclinations. In order to utilize the equation /2.17.1/ let us first examine the relationship occurring between the coordinates X, Y, Z connected with the system of astronomical coordinates α and δ and the system connected with the rotating Earth, which will be termed ξ, η, ζ /the axis ξ in the plane of the Greenwich meridian/

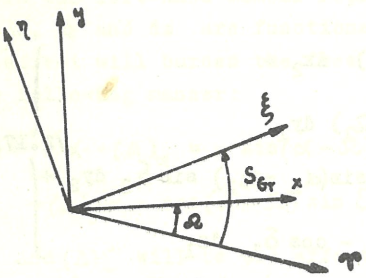


Fig.1

$$\left.\begin{aligned} X &= \xi \cdot \cos(S_{Gr} - \Omega) - \eta \cdot \sin(S_{Gr} - \Omega) \\ Y &= \xi \cdot \sin(S_{Gr} - \Omega) + \eta \cdot \cos(S_{Gr} - \Omega) \\ Z &= \zeta \end{aligned}\right\} \quad /2.22/$$

where S_{Gr} - the Greenwich sidereal time, analogically:

$$\left.\begin{aligned} dX &= d\xi \cdot \cos(S_{Gr} - \Omega) - d\eta \cdot \sin(S_{Gr} - \Omega) \\ dY &= d\xi \cdot \sin(S_{Gr} - \Omega) + d\eta \cdot \cos(S_{Gr} - \Omega) \\ dZ &= d\zeta \end{aligned}\right\} \quad /2.23/$$

Substituting it in /2.18.1/, we obtain:

$$\begin{aligned} (\Delta)\alpha_2 - (\Delta)\alpha_1 = &- \sin(\alpha_2 - \Omega_2)\left[d\xi \cdot \cos(S_{Gr2} - \Omega_2) - d\eta \cdot \sin(S_{Gr2} - \Omega_2)\right] + \\ &+ \cos(\alpha_2 - \Omega_2)\left[d\xi \cdot \sin(S_{Gr2} - \Omega_2) + d\eta \cdot \cos(S_{Gr2} - \Omega_2)\right] + \\ &+ \sin(\alpha_1 - \Omega_1)\left[d\xi \cdot \cos(S_{Gr1} - \Omega_1) - d\eta \cdot \sin(S_{Gr1} - \Omega_1)\right] + \\ &- \cos(\alpha_1 - \Omega_1)\left[d\xi \cdot \sin(S_{Gr1} - \Omega_1) + d\eta \cdot \cos(S_{Gr1} - \Omega_1)\right]; \end{aligned} \quad /2.24/$$

$$(\Delta)\alpha_2 - (\Delta)\alpha_1 = d\xi\left[-\sin(\alpha_2 - S_{Gr2}) + \sin(\alpha_1 - S_{Gr1})\right] + d\eta\left[\cos(\alpha_2 - S_{Gr2}) - \cos(\alpha_1 - S_{Gr1})\right]; \quad /2.25/$$

The condition for the expression /2.25/ to be = 0 is

$$\alpha_1 - S_{Gr1} = \alpha_2 - S_{Gr2} \quad /2.26/$$

which will be satisfied when the observations are carried out on the same hour circle, and especially in the meridian. Neither in this case, when the equation /2.17.1/ is being accepted as an observation equation, can all elements be accurately determined, because some differences of quantities close to each other appear in coefficients. Thus, the above mode of successive approximation ought to be also applied here. We should, further, define the accuracy of conditions to be satisfied for observations of the satellite in the meridian and in the equator, so as to make the above method utilizable.

Suppose, we want $\frac{(\Delta)\alpha_2 - (\Delta)\alpha_1}{\rho}$ and $\frac{(\Delta)\delta_2 - (\Delta)\delta_1}{\rho}$ to be $< 1''$

Considering $dx = dy = dz = 500$ m, $\rho = 1200$ km $\alpha - S_{Gr} = 0$

we obtain:

$$\left|(\alpha_2 - S_{Gr2}) - (\alpha_1 - S_{Gr1})\right| < 40'$$

And so, if we intend to make observations near the meridian, they have to be within the range of $\pm 20'$ of the hour angle. For the second case, considering the most unfavorable conditions $\alpha_1 - \Omega_1 = 45^\circ$, $\alpha_2 - \Omega_2 = 225^\circ$, we obtain the following limitations:

$$\frac{\delta_1 + \delta_2}{2} < 30'$$

$$|\delta_2 - \delta_1| < 1^\circ$$

By rendering the mean motion independent of errors of the station position, we also make the orbital eccentricity partially independent of these errors. This is important, for the accuracy of the orbital eccentricity is significant for the computation of the radius-vector.

According to Brouwer and Clemence [1961] , page 236, we have:

$$\left.\begin{aligned} \frac{\partial x}{\partial e} &= Hx + K\dot{x} \\ \frac{\partial y}{\partial e} &= Hy + K\dot{y} \\ \frac{\partial z}{\partial e} &= Hz + K\dot{z} \end{aligned}\right\} \qquad /2.27/$$

where: $H = -\cos v$ $\quad K = \frac{2\sin v}{n}$ $\quad$ for $e \cong 0$

The part of the derivative with the coefficient H will introduce the effect of the position errors, but the other part with the coefficients K will already be disburdened of this effect. So it seems that the determination accuracy of e will be lower than that of n , yet higher than of the remaining elements.

This seems to allow to state that - having an adequate observational program - we are able to determine two of the elements we are particularly interested in, that is, n and e - with an accuracy of the order of 10^{-6}, at least.

Chapter III

PERTURBATIONS PRODUCED BY THE EARTH'S GRAVITY FIELD

According to the recommendation of the Commission VII on Celestial Mechanics of the International Astronomical Union we are going to use the following formula for the Earth's gravity potential in the external space:

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{m=n}\left(\frac{R}{r}\right)^n P_{nm}(\sin\beta)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\right]; \qquad /3.1/$$

where: $\mu = k.M$ - the gravity constant multiplied by the Earth's mass

r - the distance from the center of mass

R - equatorial radius of the Earth

C_{nm}, S_{nm} - numerical coefficients

P_{nm} - spherical functions or Legendre's associated polynomials expressed by the general formula:

$$P_{nm}(x) = \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m}\left(\frac{1}{2^n.n} \cdot \frac{d^n(x-1)^n}{dx^n}\right); \qquad /3.2/$$

The functions P_{nm} written in explicit form to the degree 4,4 as well as to the degree 5,0 and 6,0 are presented further. According to the formula /3.1/ the notion "gravity potential" will mean the potential produced by the attraction, except for the influence of the Earth's rotation, which will be omitted.

The constants appearing in the formula /3.1/ have been many a time determined with the help of different methods. A review of these determinations will allow to ascertain to what extend they may be at present considered complete. We shall refrain from the application of weighted arithmetical means, because the evaluation of the accuracy of different inferences proves difficult. Apart from accidental errors of the material observed, also systematic errors, characteristic for a given method, come into play here. Thence the difficulty and even the impossibility of establishing weights necessary for the averaging.

As a measure of dispersion of the various results we will adopt their range, i.e. the difference between the greatest and the least of values, obtained by different methods with approximately the same theoretical exactness. The value corresponding to the middle of the range will be recognized as being the most probable.

The first of the constants appearing in Equation /3.1/ - the quantity μ denoting the Earth's mass multiplied by the Gauss gravitational constant - is of an essential importance for the present subject because, as indicated in Chapter I, it is upon it that depends the absolute accuracy in the determination of the length of the radius-vector. On the other hand, the error of the adopted value of μ bears only upon the scale of the given geometric construction /for instance, the triangulation network/, causing no local deformations. Thanks to this, it is possible - by the comparision with measurements of another type - to proceed to the determination anew, with the view of perfecting the value of μ.

For the determination of μ, the method theoretically best fitting our purpose consists in executing a direct measurement of the distance Earth-to-Moon and in establishing this way the linear dimensions of the lunar orbit. We may then derive μ from the formula:

$$\mu = \frac{n^2 \, (1+\beta)^3 \, a^3}{1 + \frac{M_M}{M_E}} \; ; \qquad /3.3/$$

where: n - the mean motion, a - the semi-major axis, β - its solar perturbation, M_M, M_E - masses of the Moon and the Earth respectively.

Such an inference is almost entirely independent of geodetic measurements on the Earth's surface, since the error of geocentric coordinates of the observing station on the Earth is relatively small as compared with the distance measured. This error may be estimated to be approximately ± 200 m, this being the very accuracy of the distance measurement itself. But a source of errors is the unsufficient knowledge of the Moon's shape in addition to a too low accuracy of the ratio M_M/M_E.

This causes that the accuracy of this method is at present still lower than of other methods. Yet, there is no doubt that within the nearest years both the figure of the Moon and the M_M/M_E ratio will be precisely determined by means of astronautical methods, and in consequence also the quantity μ. According to the method mentioned observations have been made in the United States /Yaplee and others 1959/. Their results are given and discussed by Kaula /1963 c and d/.

All other results contained in Table 1 are depending in one or another way upon the geodetic measurements carried out on the surface of the Earth. To these classical methods belongs the determination of the quantity μ from the absolute measurements of the Earth's acceleration.

The Table 1 presents two results achieved by this method: the one obtained by Kaula [1961a], based on a combined adjustment of the triangulation and of the gravimetrical data; the other one grounded on the determination of γ_e, made by Uotila [1962], a_e being computed by Kaula.

Also the method based on the measurement of the lunar parallax is to a large extent dependent on the accuracy we can produce for the dimensions of the Earth, since the above measurement consists in a determination of the Earth-to-Moon distance by measuring the parallactic angle from the known base on the Earth. Such observations were made during a couple of years at the beginning of the 20th century at Greenwich and on the Cape. They are now being reduced again by Fischer [1962], using the latest results of the triangulation which connects at present those two remote points. The result of this work is shown in Table 1, item 4. The method of occultation of stars by the Moon, given by O'Keefe and Andersen [1952], may be successfully applied to the determination of μ - if we consider the radius of the Earth to be a known quantity. Using the observations made by O'Keefe and the results obtained by Fischer concerning the dimensions of the Earth, Kaula [1963 d] defined the value of μ /Table 1, item 5/.

Also observations of artificial satellites have been used for the same purpose - on the basis of the Kepler's Law and on the assumption of the known dimensions of the Earth. The work done by Kaula [1963 a and b] in this domain, is presented in Table 1, items 6 and 7.

For comparative purposes the values of μ such as adopted for computations at research centers in United States and in Soviet Union will be given. They stand exactly in the middle of the range designated by the remaining data contained in Table 1. This seems to allow to adopt in the present work the following value of μ:

$$398603 \pm 3 \text{ km}^3 \text{ sec}^{-2}$$

As far as the values of zonal harmonics of higher degrees are concerned, undoubtedly the best ones - we may even say - the only good values have been achieved thanks to observations of artificial satellites. The reason is simple: insufficiency of observational data which could be used for the same purpose by the gravitational method and the astronomical levelling method. The comparision of possibilities of those three methods was made by the author in one of his earlier publications [Zieliński 1963] . While the gravimetrical and geodetic data always concern only certain fragments of the Earth, the observed satellite perturbations are reflecting the influence of the whole Earth's solid with its various irregularities. That is the reason for which the latest publications pertaining to geodetic data [Fischer 1961] and to gravimetrical data [Uotila 1963] adopt the values of the flattening and the harmonics C_{20}, C_{30} and C_{40}, and do not consider them to be unknowns. It does not mean, however, that the latter data are unquestionable. There exist in the satellite method many error sources which do not permit to solve this problem simply at once. We see, for instance that - when computing a certain finit number of terms - the effect of the next terms is supposed to be equal to zero this does not, however, correspond to the reality. Another discrepancy is caused by the errors of geocentric coordinates of stations and by the disturbing action of other factors, mainly atmospherical. This explains why the results given in Table2. column C_{20} differ one from another.

It can be seen from that list that all C_{20} are included within the range from 1082.2 to 1083.3, and if we omit the results obtained by Zhongolovich, based on observations of close satellites, the upper limit will amount to 1083.15.

Thereupon, we shall adopt for our computations:

$$C_{20} = - 1082.7 \pm 0.5$$

A similar review of the values C_{30} shows that they range between 2.29 - 2.59, without counting the result of Kaula [1961a] , as differring distinctly from the remaining ones and having been obtained in a way which can bring about additional errors [Zieliński 1963] . Thus, in accord with the principle adopted, we shall have:

$$C_{30} = + 2.45 \pm 0.15$$

Considering the values C_{40} we obtain /omitting the Zhongolovich's result/ the range from 1,03 to 2,1, adopting:

$$C_{40} = + 1.60 \pm 0.50$$

and for C_{50} /omitting the Kaula's result 1961/ - the range from + 0.07 to + 0.23, adopting

$$C_{50} = + 0.15 \pm 0.08$$

The present task might have been dealt with somewhat differently by placing more reliance in the latest results based on a greater number of observations; then the coefficients adopted here would certainly be nearer the reality. However, the purpose of this work is not to find the most probable values of C_{nm}, S_{nm}, but to examine their effect when the extremal possible values will be adopted. For this reason, the method applied here seems to be adequate.

As to the tesseral harmonics, the comparative data are here much more scarce, but the dispersion greater. The possibilities of the satellite method are in this case much more limited and become comparable to the possibilities of the gravimetrical method. In consequence of the Earth's rotation about its axis, the tesseral harmonics do not produce distinct - as do the zonal harmonics - long-period effects in the variations of orbital elements. The effects of errors of station coordinates and of observational irregularities appear here still more strongly. And so, in order to be able to obtain correct results, we ought to have at our disposal a richer documentation than available at present.

For calculations, data will be used - averaged in the same manner as before:

$$C_{22} = + 1.15 \pm 0.70 \qquad S_{22} = - 1.25 \pm 1.00$$

Table 1

Method	Author	f.M km^3 sec^{-2}
Radar measurement of the Earth-to-Moon distance	Yaplee and others /1963/	398605.7
Geodetic measurements	Kaula /1961/	398602.0
	Kaula+Uotila /1962/	398604.3
	Fischer /1962/	398604.0
Motion of the Moon and geodetic measurements	Fischer /1962/ + O'Keefe + Anderson	398605.7
Photographical observations of artificial satellites	Kaula /1963/1960 Iota 2	398603.7
	Kaula /1963/1961 Alpha Delta 1	398599.3
Compilation	NASA	398603.2
Compilation	Michajłow /1964/	398603

Since the attraction of the Earth is so most important of the forces acting upon the artificial satellite, the perturbations produced by the Earth's gravity field evidently belong to the greatest disturbances; hence, their qualitative and quantitative evaluation will be of a fundamental significance for the present subject matter. Such evaluation will be made by using the numerical integration method. For this purpose, we are going to use formulae for equations of satellite motion in rectangular coordinates, taking into account the farther terms of the Earth's gravitational potential up to C_{50} for zonal harmonics and to C_{22}, S_{22} - for tesseral harmonics. Those formulae were given by Kotchina [1962] in a somewhat different form without derivations; they have been derived by the author again for checking purposes /Zieliński 1967/.

Since we are interested only in short-period and diurnal perturbations, the integration will be performed over the interval of a single revolution of the satellite for zonal harmonics and over 24 hours - for tesseral harmonics.

The differential equations of satellite motion in rectangular coordinates have the form:

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{\partial U}{\partial x} \\ \frac{d^2y}{dt^2} &= \frac{\partial U}{\partial y} \\ \frac{d^2z}{dt^2} &= \frac{\partial U}{\partial z} \end{aligned}\right\} \qquad /3.4/$$

where U is the gravitational potential defined by the formula /3.1/. We write it now in an explicit form, confining ourselves to the terms 50 in zonal harmonics and to 22 - in tesseral ones /and remembering that C_{10}, C_{11}, S_{11}, C_{21}, S_{21} are equal to 0/:

$$U = U_{00} + U_{20} + U_{22} + U_{30} + U_{40} + U_{50}$$

where the particular terms are respectively:

$$U_{00} = \frac{\mu}{r} \qquad /3.5/$$

$$\left.\begin{aligned}
U_{20} &= \frac{\mu}{r}\left(\frac{R}{r}\right)^2 C_{20} \cdot P_{20}(\sin\beta)\\
U_{22} &= \frac{\mu}{r}\left(\frac{R}{r}\right)^2 \left(C_{22}\cos 2\lambda + S_{22}\sin 2\lambda\right) \cdot P_{22}(\sin\beta)\\
U_{30} &= \frac{\mu}{r}\left(\frac{R}{r}\right)^3 C_{30} \cdot P_{30}(\sin\beta)\\
U_{40} &= \frac{\mu}{r}\left(\frac{R}{r}\right)^4 C_{40} \cdot P_{40}(\sin\beta)\\
U_{50} &= \frac{\mu}{r}\left(\frac{R}{r}\right)^5 C_{50} \cdot P_{50}(\sin\beta)
\end{aligned}\right\} \qquad /3.5/$$

Let us now introduce the coordinate system x,y,z, the origin of which is at the center of the Earth's mass, the z-axis coincides with the rotation axis, the x-axis being orientated in the direction of the vernal equinox. After expressing functions P_{nm} by x,y,z , and $\sin 2\lambda$, $\cos 2\lambda$ by sidereal time - s, and after the differentiation, we have:

$$\left.\begin{aligned}
\frac{\partial U_{00}}{\partial x} &= -\frac{\mu x}{r^3}\\
\frac{\partial U_{20}}{\partial x} &= \frac{3}{2}\mu R^2 C_{20}\left(\frac{-5xz^2}{r^7} + \frac{x}{r^5}\right)\\
\frac{\partial U_{30}}{\partial x} &= \frac{5}{2}\mu R^3 C_{30}\left(\frac{-7xz^3}{r^9} + \frac{3xz}{r^7}\right)\\
\frac{\partial U_{40}}{\partial x} &= \frac{15}{8}\mu R^4 C_{40}\left(\frac{-21xz^4}{r^{11}} + \frac{14xz^2}{r^9} - \frac{x}{r^7}\right)\\
\frac{\partial U_{50}}{\partial x} &= \frac{21}{8}\mu R^5 C_{50}\left(\frac{-33xz^5}{r^{13}} + \frac{30xz^3}{r^{11}} - \frac{5xz}{r^9}\right)\\
\frac{\partial U_{60}}{\partial x} &= \frac{7}{16}\mu R^6 C_{60}\left(\frac{-429xz^6}{r^{15}} + \frac{495xz^4}{r^{13}} - \frac{135xz^2}{r^{11}} + \frac{5x}{r^9}\right)
\end{aligned}\right\} \qquad /3.6/$$

$$\left.\begin{aligned}
\frac{\partial U_{00}}{\partial y} &= -\frac{\mu y}{r^3}\\
\frac{\partial U_{20}}{\partial y} &= \frac{3}{2}\mu R^2 C_{20}\left(\frac{-5yz^2}{r^7} + \frac{y}{r^5}\right)\\
\frac{\partial U_{30}}{\partial y} &= \frac{5}{2}\mu R^3 C_{30}\left(\frac{-7yz^3}{r^9} + \frac{3yz}{r^7}\right)\\
\frac{\partial U_{40}}{\partial y} &= \frac{15}{8}\mu R^4 C_{40}\left(\frac{-21yz^4}{r^{11}} + \frac{14yz^2}{r^9} - \frac{y}{r^7}\right)\\
\frac{\partial U_{50}}{\partial y} &= \frac{21}{8}\mu R^5 C_{50}\left(\frac{-33yz^5}{r^{13}} + \frac{30yz^3}{r^{11}} - \frac{5yz}{r^9}\right)
\end{aligned}\right\} \qquad /3.7/$$

$$\left.\begin{aligned}
\frac{\partial U_{00}}{\partial z} &= -\frac{\mu z}{r^3}\\
\frac{\partial U_{20}}{\partial z} &= \frac{3}{2}\mu R^2 C_{20}\left(\frac{3z}{r^5} - \frac{5z^3}{r^7}\right)
\end{aligned}\right\} \qquad /3.8/$$

$$\left.\begin{aligned}
\frac{\partial U_{30}}{\partial z} &= \frac{1}{2}\mu R^3 C_{30}\left(\frac{-35z^4}{r^9} + \frac{30z^2}{r^7} - \frac{3}{r^5}\right)\\
\frac{\partial U_{40}}{\partial z} &= \frac{5}{8}\mu R^4 C_{40}\left(\frac{-63z^5}{r^{11}} + \frac{70z^3}{r^9} - \frac{15z}{r^7}\right)\\
\frac{\partial U_{50}}{\partial z} &= \frac{3}{8}\mu R^5 C_{50}\left(\frac{-231z^6}{r^{13}} + \frac{315z^4}{r^{11}} - \frac{105z^2}{r^9} + \frac{5}{r^7}\right)
\end{aligned}\right\} \quad /3.8/$$

The derivatives of tesseral harmonics may be represented by the general formula:

$$\frac{\partial U_{nm}}{\partial q} = k_{nm}\,\mu R^n\left[\cos ms\left(Pq_{nm}\,C_{nm} + Qq_{nm}\,S_{nm}\right) + \sin ms\left(Qq_{nm}\,C_{nm} - Pq_{nm}\,S_{nm}\right)\right] \quad /3.9/$$

where: q is identical to x or y or z.

The value of k_{nm} for $n = 2,\ m = 2$ is: $k_{22} = 3$

whereas the respective Pq_{nm} and Qq_{nm} :

$$\left.\begin{aligned}
Px_{22} &= \frac{-5/x^2-y^2/x}{r^7} + \frac{2x}{r^5}\ ; & Qx_{22} &= \frac{-10x^2y}{r^7} + \frac{2y}{r^5}\\
Py_{22} &= \frac{-5y/x^2-y^2/}{r^7} - \frac{2y}{r^5}\ ; & Qy_{22} &= \frac{-10\ xy^2}{r^7} + \frac{2x}{r^5}\ ;\\
Pz_{22} &= \frac{-5z/x^2-y^2/}{r^7}\ ; & Qz_{22} &= \frac{-10xyz}{r^7}\ ;
\end{aligned}\right\} \quad /3.10/$$

The numerical methods have gained in significance especially after the introduction of electronic computers. A number of new methods have arisen, adapted to the new possibilities, optimalizing the processes of calculus and programming. Among them is the method given by C. Runge in 1895, improved by W. Kutta in 1901 and adapted to the mechanical computation by S. Gill in 1951. A full description of the Runge-Kutta-Gill method, including the derivation of formulae and the logical scheme of the program is presented in the Romanelli's work [1962]. Those formulae are constituting a simple computation scheme. Suppose that we have $n + 1$ differential equations of the first order in the form:

$$y_i'(x) = f_i\left(y_0\ (x)\ ,\ y_1(x)\ldots\ldots\ldots\ldots y_n\ (x)\right) \quad /3.11/$$

whereat:

$$y_0'(x) = f_0 = 1 \text{ or } y_0\ (x) = x \quad /3.12/$$

and boundary values

$$y_i\ (x_0) = y_{io} \quad /3.13/$$

The indice j will denote the number of the iteration, that is:

$$j = 1,\ 2,\ 3,\ 4$$

For $i = 0,\ 1,\ 2,\ \ldots\ldots\ldots n$, we calculate:

$$y_{ij}' = k_{ij} = f_i\left(y_{o,j-1},\ y_{1,j-1}\ \ldots\ldots\ y_{j,n-1}\right) \quad /3.14/$$

For $j = 1$; $y_{i,o} = y_i\ (x_o)$ so this is either the initial value /for the first step/ or y_i computed in the preceding step.

Next:

$$y_{ij} = y_{i,j-1} + h\left[a_j\left(k_{ij} - b_j\ q_{i,j-1}\right)\right] \quad /3.15/$$

$$q_{ij} = q_{i,j-1} + 3\left[a_j\left(k_{ij} - b_j\ q_{i,j-1}\right)\right] - c_j \cdot k_{ij} \quad /3.16/$$

where:

Table 2

The coefficientus of the Earth gravity field

Author /year/ the method applied	C_{20}	C_{30}	C_{40}	C_{50}	C_{22}	S_{22}
O'Keefe, Eckels, Squires /1959/ Vanguard I	-1082.5	+2.4	+1.7	+0.1	-	-
King-Hele /1961/ Sputnik 2, Vanguard 1, Explorer 7	-1082.79	-	+1.4	-	-	-
Zhongolovich /1960/ Sputnik 2, Sputnik 3, Sputnik 3 R	-1083.3	+2	+4.1	-	-	-
Kozai /1961/ Explorer 7, Vanguard 3, Vanguard 1	-1082.21	+2.29	+2.1	+0.23	-	-
Michielsen /1961/ Transit 1B, Vanguard 1, Sputnik 4	-1082.7	+2.4	+1.7	+0.1	-	-
Newton, Hopfield, Kline/1961/ Transit 2A, Vanguard 1, Transit 1B	-	+2.36	-	+0,19	-	-
Smith /1961/ Sputnik 3, Vanguard 2, Transit 1B, Tiros 1	-1083.15	-	+1.4	-	-	-
Shelkey /1962/	-1082.61	-	+1.52	-	-	-
Kozai /1962/ 15 satellites	-1082.48	+2.562	+1.84	+0.064	-	-
Kozai /1964/ 8 satellites	-1082.65	+2.53	+1.62	+0.21	-	-
Smith /1963/	-	+2.44	-	+0.18	-	-
King Hele, Cook, Rees /1963/ 7 satellites	-1082.86	-	+1.03	-	-	-
Kaula /1961/ satellites + gravimetry + triangulation	-1082.61	+2.05	+1.43	-0.08	+0.48	-0.25
Kaula /1963/ 1959 α 1, 1959 η, 1960 ι 2	-1082.49	+2.59	+1.65	+0.10	+1.19	-1.10
Kaula /1963/ 1959 α 1, 1959 η, 1960 ι 2, 1961 δ 1, 1961 α δ 1	-1082.44	+2.57	+2.01	+0.07	+1.21	-0.89
Uotila /1962/ gravimetry	-	-	-	-	+0.45	-1.45
Kozai /1961/ Sputnik 3, Vanguard 1, Vanguard 3	-	-	-	-	+0.60	-2.24
Kozai /1963/	-	-	-	-	+0.72	-0.95
Izsak /1963/ Vanguard 2, Vanguard 3, Explorer 9 Echo 1R, Midas 4	-	-	-	-	+0.968	-0.40
Anderle, Oesterwinter /1963/ Anna Ib	-1082.466	+2.476	+1.405	+0.140	+1.84	-0.99
Guier /1963/	-	-	-	-	+1.68	-0.64
Izsak /1964/ 8 satellites	-	-	-	-	+0.75	-0.61

$$
\left.\begin{array}{lll}
a_1 = \frac{1}{2} & b_1 = 2 & c_1 = \frac{1}{2} \\
a_2 = 1 - \sqrt{\frac{1}{2}} & b_2 = 1 & c_2 = 1 - \sqrt{\frac{1}{2}} \\
a_3 = 1 + \sqrt{\frac{1}{2}} & b_3 = 1 & c_3 = 1 + \sqrt{\frac{1}{2}} \\
a_4 = \frac{1}{6} & b_4 = 2 & c_4 = \frac{1}{2}
\end{array}\right\} \qquad /3.17/
$$

h - value of the step.

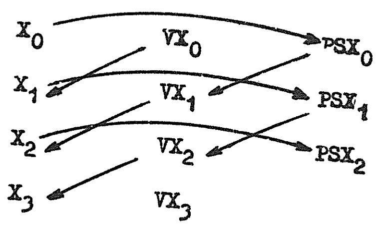
We iterate the calculation cycle /3.14/, /3.15/, /3.16/ for $j = 2, 3, 4$. Having the values of $y_{i,4}$ we are proceeding to the next step, substituting them for $y_{i,o}$.

In our case, we have to integrate the following equations:

$$
\left.\begin{array}{l}
k_0 = \frac{dt}{dt} = 1 \\
k_1 = \frac{dx}{dt} = VX \\
k_2 = \frac{dy}{dt} = VY \\
k_3 = \frac{dz}{dt} = VZ \\
k_4 = \frac{d^2x}{dt^2} = \frac{dVX}{dt} = PSX \\
k_5 = \frac{d^2y}{dt^2} = \frac{dVY}{dt} = PSY \\
k_6 = \frac{d^2z}{dt^2} = \frac{dVZ}{dt} = PSZ
\end{array}\right\} \qquad /3.18/
$$

Initial data: X_0, Y_0, Z_0, VX_0, VY_0, VZ_0.

Since the motion equation is an equation of the second order, it has to be integrated twice; the first equation will give the velocity, the second one - the coordinates. This is shown in the scheme:



Yet, not the coordinates by themselves are of interest for our subject matter, but the radius-vector or, more precisely, the variations of it produced by one or another term of the formula for the gravity potential. On account of this, the program has been elaborated so as to perform the entire integration cycle twice for the two pairs of constants C'_{nm}, S'_{nm} and C''_{nm}, S''_{nm}, whereas only the result:

$$\Delta R = R'' - R'$$

was made available in outlet, where: /3.19/

$$R = \sqrt{X^2 + Y^2 + Z^2} \qquad /3.20/$$

For C'_{nm}, S'_{nm}, C''_{nm}, S''_{nm} were either substituted the values of harmonics differing one from another by the error value according to the definition given earlier in this Chapter, or 0 was substitutes for C', S', and the most probable values - for C'', S''. In this manner, both the absolute quantities of perturbations and their possible errors had been calculated. In addition to ΔR, the variations of the latitude argument Δu have been computed. They are useful for an estimation as regards perturbations to be taken into account in some methods described in Chapter V, in which we are using the increment of the latitude argument during a short time interval.

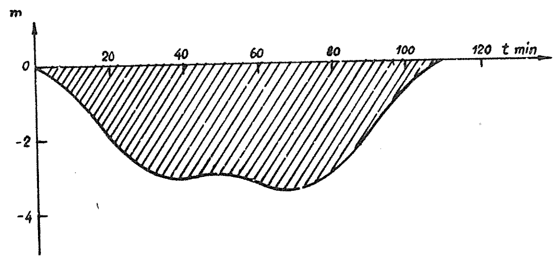


Fig.2

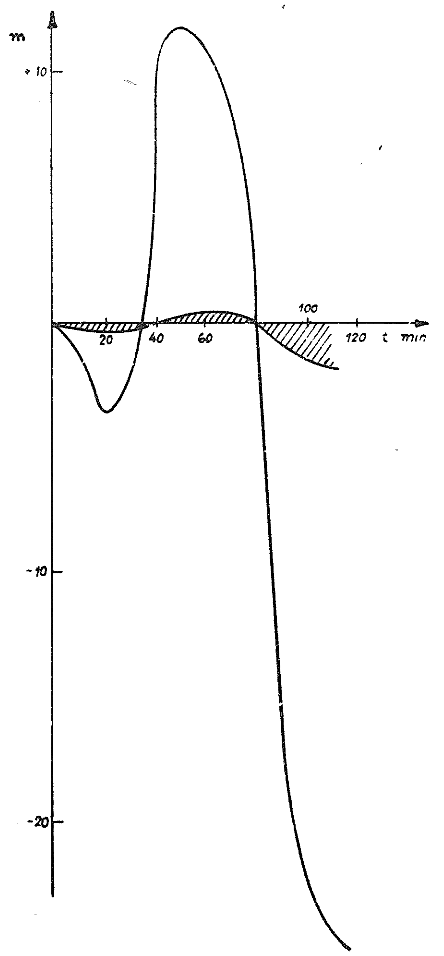


Fig.3

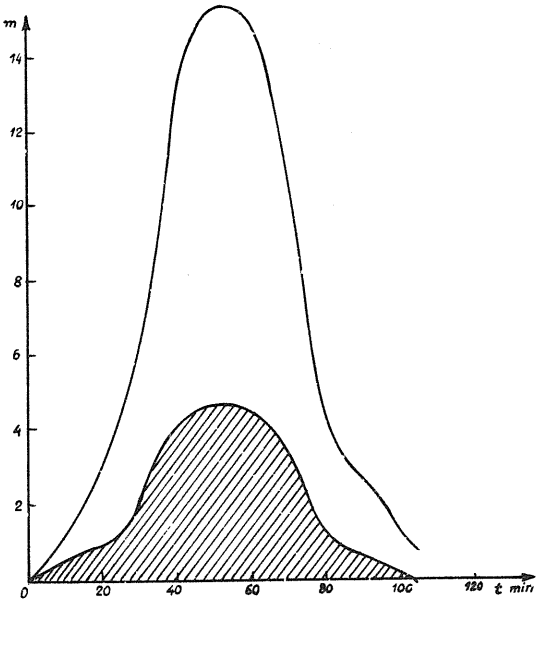


Fig.4

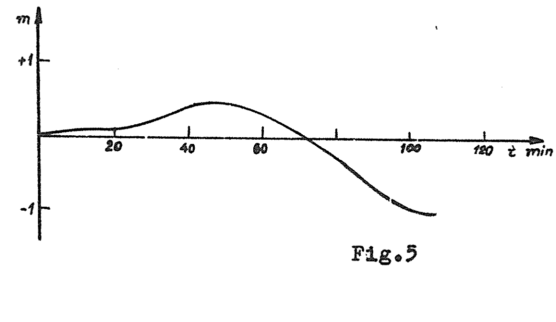


Fig.5

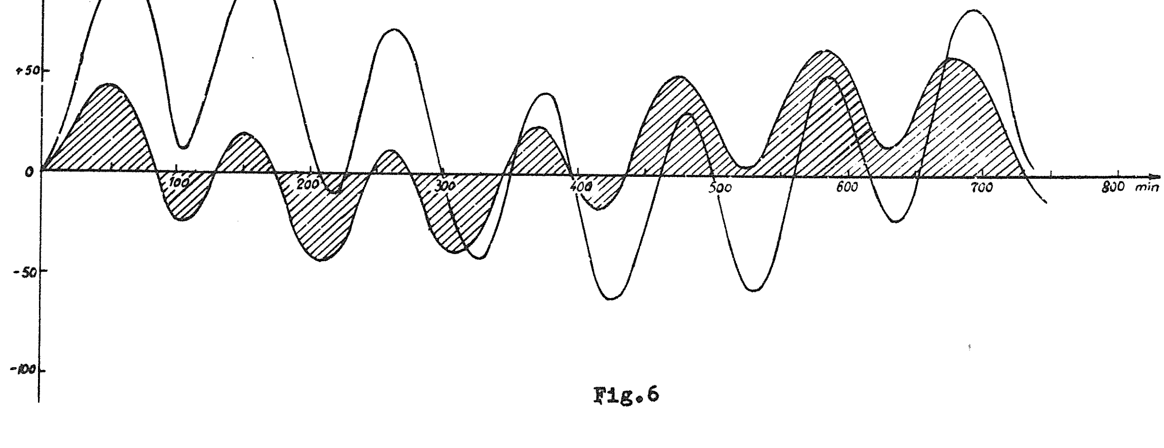


Fig.6

The method adopted here for computing ΔR has one more essential quality: it eli
almost entirely the integration errors. The integration error increasing with the proce
calculus, has its source in the very principle of the numerical integration: emission o
of higher orders, computation with a finite number of significant digits and roundings,
tions applied in every step. The occurance of the integration errors is not thoroughly e
and not always is it possible to predict their magnitude, the more so as they depend als
the computer employed. The error is supposed to be proportional to the number of steps i
power $\frac{3}{2}$ [Ziółkowski 1964] and to have at the same time an oscillating run, increasing
the function is rapidly changing and decreasing where the variations are proceeding slowl
examination purposes, a special program, the RKG-TEST, had been established where the int
tion results of the Keplerian motion were compared with calculations carried out after th
formulae for this motion. It appeared that after 250 steps the differences arise at the s
significant digit, whereas in the radius-vector the difference amounts only to two units a
seventh digit. If the radius itself is burdened with a small error, the difference of two
will practically be acquitted of integration errors within the limits of the required numb
steps /maximally 1440/. It has been adopted that the integration step is equal to one side
minuts, this corresponding approximately to 1/100 of the revolution period. On account of
circular orbit, the constant integration step has been accepted.

A few words should also be devoted to the characteristics of the computer with the he
of which the calculus has been carried out.

It is a GIER machine made in Danemark executing some 10.000 floating-point operations
second. Calculations are carried out with an accuracy to 29 digits in the binary system, th
corresponding to approximately $8\frac{1}{2}$ decimal digits, that means that the roundings appear whe
number is greater than $2^{29} = 536\ 870\ 912$. The programming is performed in the ALGOL 60 langu
so it is relatively easy to be mastered. The programs are universal and may be used by other
at different computation centers. The copies of programs are enclosed to the mentioned work
[Zieliński 1967].

For computations, the orbit of Alouette from June 1963 was adopted: $i = 80^{\circ}\!.5$; $e = 0.00$
$P = 105^{m}\!.4$; $a = 7.392$ Mgm, this corresponding to $H_p = 996$ km. It is the lowest orbit within th
frame of the limitations adopted, thus the perturbations caused by the Earth's gravity field
will be the most sizable. Since the orientation of the orbit in space, except for i, does not
play a part here, we have adopted, for simplification: $\Omega = 0$, $\omega = 0$, $M_o = 0$. For simplifi-
cation and shortening the computations, the equatorial radius of the Earth $R = 6.378165$ Mgm
has been adopted as the length unit. In this manner, one could avoid the raising to a high po-
wer of R appearing in equations. The adoption of the time unit equal to 1 min allowed as well
to avoid a certain number of operations. In this connection, the constant μ and the coordi-
nates and the initial velocities have assumed the following values:

$$\mu = 0.0055002277\ R^3/\ min^{*2}$$
$$X_0 = 1.15594871\ R$$
$$Y_0 = 0$$
$$Z_0 = 0$$
$$VX_0 = 0$$
$$VY_0 = 0.0113991512\ R/min^{*}$$
$$VZ_0 = 0.0681186354\ R/min^{*}$$

The conversion into meters again was performed just before deriving the result.

The integration outcomes are presented in the form of diagrams. The computations were made according to the following sequance:

Variant no 1: effect of error of the term C_{20} amounting to

$$\Delta C_{20} = 5.10^{-7}\ /Fig.2/\ ;$$

Variant no 2: effect of the term $C_{30} = +\ 245.10^{-8}$ /Fig.3/ ;
Variant no 3: effect of error of the term C_{30} : $\Delta C_{30} = 15.10^{-8}$ /Fig.3/ ;
Variant no 4: effect of the term $C_{40} = +\ 16.10^{-7}$ /Fig.4/ ;
Variant no 5: effect of error of the term C_{40} : $\Delta C_{40} = 5.10^{-7}$ /Fig.4/ ;
Variant no 6: effect of the term $C_{50} = +\ 15.10^{-8}$ /Fig.5/ ;
Variant no 7: effect of the terms $C_{22} = +\ 115.10^{-8}$ and $S_{22} = -\ 125.10^{-8}$ /Fig.6/ ;
Variant no 8: effect of error of the terms C_{AA} and S_{22} amounting to $\Delta C_{22} = 70.10^{-22}$
$\Delta S_{22} = 100.10^{-8}$ /Fig.6/.

For reasons beyond the author's control, there was no possibility of cyrrying out the further calculus. The effect of the terms C_{3m}, S_{3m} had not been computed. This will certainly be done in the future, for the program is already prepared. At the moment we may, however, say that it is supposed to be smaller than the effect of C_{22} , S_{22} .

From the obtained results the following conclusions are proceeding: the accuracy known for the coefficients of zonal harmonics allows for the radius-vector to be computed with an accuracy of 10^{-6}. For this purpose, the terms up to C_{40} ought to be taken into account, the accuraties known for them being adequate. As regards the perturbations produced by harmonics C_{22}, S_{22}, they exceed 100 m, so they have to be taken into consideration. Simultaneously, however, the accuracy of coefficients, defined in accord with the principle adopted in the preceding chapter, will produce an error in the determination of the length of the radius-vector, amounting to 70 m, this given an accuracy of 10^{-5}.

Chapter IV

OTHER PERTURBATIONS

1. Atmospheric Drag

The atmospheric drag can be fairly called the major ennemy of geodetic satellites. It is for this reason that we have already at the beginning adopted assumptions tending to minimize the perturbations caused by the resistance of the medium. And so, we have adopted an orbit approximate to the circular one, at an altitude of more than 1000 km over the Earth's surface, and a satellite of a small area-to-mass ratio /the so-called heavy satellite of the type of Anna 1 B or Alouette/. It might be said, in addition, that in our case only the short-period and diurnal perturbations seem to be dangerous because they do not find reflection in variations of the mean elements which can be determined from observations. We shall now try to estimate the order of magnitude of those perturbations and to find out whether they can be regarded as negligibly small.

Let us briefly recall the mechanism of the arising of perturbations produced by the atmospheric drag. Suppose that the atmosphere remains motionless with regard to the orbital plane whose eccentricity $\neq 0$. Moving in a medium offering resistance, the satellite consumes a part of its kinetic energy and loses by the same its velocity. Since the greatest retardation occurs in the vinicity of the perigee, it is there that the largest velocity variations take place, this producing a change in the shape and in the dimensions of the orbit; it should be added that the distance to the perigee will show markedly slower variations than the distance to the apogee. Thus we shall have secular variations of two elements of the semi-axis a, and in consequence of the period P and the mean motion n , and of the eccentricity e. Still, in addition to these secular variations leading to the circular shape taken by the orbit and to the lowering of the altitude at which the satellite is moving until it collapses, short-period variations may be expected to occur, for at each point of the orbit the retardation force has a different magnitude which can be represented as a function of anomaly. It is only in the case of an orbit being circular and the atmosphere having a spherical structure that the periodical terms in perturbations of elements could be avoided. Unfortunately, the atmosphere does not represent an ideal spheroidal structure. As demonstrated by explorations based on observations of artificial satellites, the atmosphere has rather an ovoid shape produced by the difference in temperatures of the daytime and the night-time side of the Earth globe. The axis of the bulge forms an angle of near 30° in right ascension with the Earth-to-Sun direction and is orientated eastward from it. Moreover, owing to the rotation about the Earth's axis, a flattening of the atmosphere can be noticed. All that causes that even a satellite being in circular orbit, is moving in a medium whose density is a function of the place and so, short-period perturbations may occur.

If a great number of studies is devoted to the problem concerning long-period and secular variations, the theory of their computation, the determination, the atmospheric parameters from observations, rather a small number of authors have dealt with the short-period variations. This problem has been approached in a more detailed manner by Brouwer and Hori [1961] and also by Izsak [1960] . In both cases though concrete, but rather complicated, models of the atmosphere have been adopted, this causing that after the integration of differential perturbation equations, very complicated formulae had been obtained. For our purposes, the formulae given by Batrakov and Proskurin [1959] shall be used:

$$\left.\begin{aligned}
\frac{da}{dt} &= -\frac{2\alpha\varrho\ n\ a^2}{(1-e^2)^{3/2}}\left(1+2\ e\cos v+e^2\right)^{3/2}\\
\frac{de}{dt} &= -\frac{2\alpha\varrho\ n\ a}{(1-e^2)^{1/2}}\left(1+2\ e\cos v+e^2\right)^{1/2}(e+\cos v)\\
\frac{d\Omega}{dt} &= \frac{di}{dt} = 0\\
e\ \frac{d\omega}{dt} &= -\frac{2\alpha\varrho\ n\ a}{(1-e^2)^{1/2}}\left(1+2\ e\cos v+e^2\right)^{1/2}\sin v\\
\frac{d\varepsilon}{dt} &= \frac{2\alpha\varrho\ n\ a\ e}{(1-e^2)^{1/2}}\left(1+2\ e\cos v+e^2\right)^{1/2}\left(\frac{(1-e^2)^{1/2}}{1+e\cos v}+\frac{1}{1+\sqrt{1-e^2}}\right)\sin v
\end{aligned}\right\}\quad /4.1/$$

where

$$\alpha = \frac{1}{2}\ C_x\ \frac{S}{m}$$

ϱ = density of atmosphere,
C_x = aerodynamic coefficient which amounts to 2 for this type of motion,
S = area of the cross-section of satellite,
m = mass of satellite,
ε = longitude in orbit.

The equations /4.1/ do not contain any assumptions pertaining to the law of variability of ϱ , according to the place. The only assumption introduced is the neglecting of the rotary motion of the atmosphere, owing to this, $d\Omega = di = 0$. In our case, where the question is of examining the order of magnitude rather than an accurate determination of perturbations - and when small quantities are to be expected - we may allow ourselves far-going simplifications.

Let us assume a circular orbit in the equatorial plane: $e = 0$, $i = 0$, $a = 7378$ km. Thus:

$$v = M = n.t \qquad /4.2/$$

and the formulae /4.1/ will have the form:

$$\begin{aligned}
\delta a &= -2\alpha\ n\ a^2 \cdot \int\varrho\ dt\\
\delta e &- 2\quad n\ a \cdot \int\cos M\ \varrho\ dt\\
\delta\varepsilon &= 0
\end{aligned}\qquad /4.3/$$

$\delta\omega$ for $e = 0$ takes indeterminate values, so we have not to take it into consideration.

Further, let the satellite be of the shape and of the dimensions of Alouette:

$$S = 0.735\ m^2, \qquad m = 145\ kg$$

Therefore:

$$\alpha = 0.0051\ m^2/kg = 0.051\ cm^2/g$$

$$n^2a^2 = \mu/a = 398603\ km^3sec^{-2}/7378\ km = 54.0259\ km^2/sec^2$$

$$na = 7.35023\ km/sec = 7.35023 \cdot 10^5\ cm/sec$$

$$n = 0.0009962\ sec^{-1}$$

$$na^2 = 54230\ km^2/sec = 5.4230 \cdot 10^{14}\ cm^2/sec$$

$$2\alpha\ n\ a^2 = 0.553 \cdot 10^{14}\ cm^4/g \cdot sec$$

$$2\alpha\ n\ a = 0.750 \cdot 10^5\ cm^3/g \cdot sec$$

In order to compute the value of the integral $\int\varrho\ dt$, we are going to use the model of atmosphere given by Martin and others [1961] . It is a model based on observations of artificial satellite, which takes into account the daytime bulge, upon assumption of the mean level

Table 3

Δα	θ	q(θ)	q(θ).(log ϱ_max − log ϱ_min)	log ϱ	ϱ (g/cm³)
1	2	3	4	5	6
180°	0^h	0.183	0.258	$\bar{18}.678$	$4.76\cdot10^{-18}$
210	2	0.106	0.149	$\bar{18}.570$	$3.72\cdot10^{-18}$
240	4	0.030	0.042	$\bar{18}.463$	$2.90\cdot10^{-18}$
260	5^h20^m	0.000	0.000	$\bar{18}.421$	$2.64\cdot10^{-18}$
270	6	0.020	0.028	$\bar{18}.449$	$2.81\cdot10^{-18}$
300	8	0.334	0.470	$\bar{18}.891$	$7.78\cdot10^{-18}$
330	10	0.728	1.024	$\bar{17}.445$	$2.79\cdot10^{-17}$
0	12	0.942	1.324	$\bar{17}.755$	$5.69\cdot10^{-17}$
30	14	1.000	1.406	$\bar{17}.827$	$6.72\cdot10^{-17}$
60	16	0.935	1.315	$\bar{17}.736$	$5.44\cdot10^{-17}$
90	18	0.724	1.018	$\bar{17}.439$	$2.75\cdot10^{-17}$
120	20	0.463	0.651	$\bar{17}.072$	$1.18\cdot10^{-17}$
150	22	0.276	0.388	$\bar{18}.809$	$6.44\cdot10^{-18}$
180	24	0.183	0.257	$\bar{18}.678$	$4.76\cdot10^{-18}$

Table 4

Δα	Δt sec	ϱ·Δt g.cm⁻³.sec	Δa cm	cos M	ϱ.Δt.cos M	Δe
1	2	3	4	5	6	7
180°						
	263	$1.25\cdot10^{-15}$	−0.07	1.000	$1.25\cdot10^{-15}$	$-1\cdot10^{-10}$
210						
	526	$1.96\cdot10^{-15}$	−0.18	0.866	$1.70\cdot10^{-15}$	$-2\cdot10^{-10}$
240						
	438	$1.27\cdot10^{-15}$	−0.25	0.500	$0.64\cdot10^{-15}$	$-3\cdot10^{-10}$
260						
	263	$0.69\cdot10^{-15}$	−0.29	0.174	$0.12\cdot10^{-15}$	$-3\cdot10^{-10}$
270						
	350	$0.98\cdot10^{-15}$	−0.34	0.000	0.00	$-3\cdot10^{-10}$
300						
	526	$4.09\cdot10^{-15}$	−0.57	−0.500	$-2.04\cdot10^{-15}$	$-1\cdot10^{-10}$
330						
	525	$14.66\cdot10^{-15}$	−1.38	−0.866	$-12.70\cdot10^{-15}$	$+8\cdot10^{-10}$
0						
	526	$29.90\cdot10^{-15}$	−3.03	−1.000	$-29.90\cdot10^{-15}$	$+31\cdot10^{-10}$
30						
	525	$35.31\cdot10^{-15}$	−4.98	−0.866	$-30.58\cdot10^{-15}$	$+54\cdot10^{-10}$
60						
	526	$28.59\cdot10^{-15}$	−6.56	−0.500	$-14.30\cdot10^{-15}$	$+64\cdot10^{-10}$
90						
	525	$14.45\cdot10^{-15}$	−7.36	0.000	0.00	$+64\cdot10^{-10}$
120						
	526	$6.20\cdot10^{-15}$	−7.71	0.500	$3.10\cdot10^{-15}$	$+62\cdot10^{-10}$
150						
	525	$3.38\cdot10^{-15}$	−7.89	0.866	$2.93\cdot10^{-15}$	$+60\cdot10^{-10}$
180						
	263	$1.25\cdot10^{-15}$	−7.96	1.000	$1.25\cdot10^{-15}$	$+59\cdot10^{-10}$

P = 6307

of solar activity. It is presented in the form of a table containing ϱ min and ϱ max for the given altitude, and of a formula for calculating ϱ at an arbitrary place. This model is certainly not an ideal one, yet possesses the quality of being relatively simple and convenient in application. Besides, it agrees quite well with other, more recent and more complicated models.

The model presented by Martin and others gives for the altitude of 1000 km:

$$\varrho_{max} = 6.72 \times 10^{-17} \text{ g/cm}^3 \qquad \varrho_{min} = 2.64 \times 10^{-18} \text{ g/cm}^3$$

Another model formulated by Paetzold and Zschröner [1961] gives respectively:

$$\varrho_{max} = 9.701 \times 10^{-17} \text{ g/cm}^3 \qquad \varrho_{min} = 4.313 \times 10^{-18} \text{ g/cm}^3$$

The Nicolet-Jacchia [Jacchia 1964] model considering many parameters, gives:

$$\varrho_{max} = 1.97 \cdot 10^{-17} \qquad \varrho_{min} = 5.86 \cdot 10^{-18}$$

assuming that the coefficient of solar activity has in the 10.8 -centimeterband the following average value:

$$F_{10.8} = 200$$

So it can be seen that the model of Martin and others may be accepted without fear of making a mistake of an order of magnitude. Martin gives the following formulae for the computation of ϱ :

$$\log \varrho (h,\theta) = \log \varrho_{min} (h) + q (\theta) \cdot \left[\log \varrho_{max} (h) - \log \varrho_{min} (h)\right] \qquad /4.4/$$

where:

h - altitude above the Earth's surface

θ - real solar time.

The values of $q(\theta)$ are presented in Table 3 col.3.

Substituting the respective values in the equation /4.4/, we obtain the values of ϱ ($\cdot$ 1000.θ) , as shown in Table 3, col.6.

$\Delta \alpha$ - stands for the difference of the right ascension between the Sun and the given point in the space; $\log \varrho_{max} = 0.827-17$ $\log \varrho_{min} = 0.421 - 18$ $\log \varrho_{max} - \log \varrho_{min} = 1.406$.

The integration will be made in the way of approximation, by the summation of products of the density ϱ multiplied by the corresponding time interval. We shall observe the motion of the satellite starting with the point lying opposite to the Sun, on the other side of the Earth: / $\Delta \alpha = 180^\circ$/.

As seen from the Table 4, the explored perturbations are very small. The secular perturbations of the semi-major axis amount to a few centimeters during one revolution, the periodic variations being fully negligible. The same is to be said in respect of the eccentricity, where the variations appear at the ninth digit after the point. This seems to entitle us to state that in the present subject matter, with so selected orbital and aerodynamic conditions, the short-period and diurnal perturbations produced by the atmospheric drag are practically of no significance.

2. Effect of the Solar and Lunar Attraction

The influence of the Sun and the Moon on the motion of the artificial satellite are differring each other only as a matter of quantity, while qualitatively they are the same. Let us first consider the influence of the Moon. We shall make here use of the formulae for the perturbation function of the exterior body, known from the celestial mechanics:

$$R = \mu' \left(\frac{1}{|\bar{r} - \bar{r}'|} - \frac{\bar{r}' \cdot \bar{r}}{r'^3} \right) \qquad /4.5/$$

where: μ' - the mass of the disturbing body multiplied by the gravitational constant,

$\bar{r}$ - the radius-vector of satellite,

$\bar{r}'$ - the radius-vector of the disturbing body

Expanding $\frac{1}{|\bar{r}\, . \bar{r}'|}$ in series of Legendre's functions and remembering that

$$\bar{r}\ .\ \bar{r}' = r\ .\ r' \cos\Psi$$

we obtain

$$R = \frac{\mu'}{r'}\left[1 + \frac{r}{r'}\ \cos\Psi\ + \frac{1}{2}\left(\frac{r}{r'}\right)^2(3\cos^2\Psi - 1) - \frac{r\, r'}{r'^2}\ \cos\Psi\right] = \frac{\mu'}{r'}\left[1 + \frac{1}{2}\left(\frac{r}{r'}\right)^2(3\cos 2\Psi - 1)\right] \qquad /4.6/$$

with an accuracy to the terms of the second order.

In the formula /4.6/ only the second term will produce short-period perturbations, because of being the function of the angle Ψ . Let us make an evaluation of the magnitude of this term.

In the case of the Moon μ' amounts to 1/81 of the terrestrial μ, $r' = 380\,000$ km, r - will be adopted for 9500 km /the altitude of the satellite orbit being approximately 3000 km/.

The ratio $\frac{r}{r'}$ will be 1/40.

Therefore:

$$R_M = \frac{\mu}{81.40.r}\left[1 + \frac{1}{2}\ \left(\frac{1}{40}\right)^2\ (3\cos 2\Psi - 1)\right]$$

The spherical function P_{20} taking the values in the range $\pm$ 1, the variable part of the formula /6/ will assume values in the range

$$\frac{\mu}{r}\ .\ 2\ .\ 10^{-7}$$

For the Sun, $\mu' = 332\,000\ \mu$, and $r' = 15\,000$ r.

Thus it will be:

$$R_M = \frac{332}{15}\ \frac{\mu}{r}\left[1 + \frac{1}{2}\left(\frac{1}{1500}\right)^2\ (3\cos 2\Psi - 1)\right]$$

the varying part of the formula being embodied in the range:

$$\frac{\mu}{r}\ .\ 22\ .\ \frac{1}{225.10^6}\ = \frac{\mu}{r}\ 1.10^{-7}$$

Both of these magnitudes are comparable with the effect of the error found for the harmonic C_{20}. As remembered from the Chapter III, this error, whose value was assumed to be 5.10^{-7}, gave us perturbations of the order of one meter. Since we have to do in this case with a still smaller quantity, it can be concluded that the solar and lunar effects need not be taken into account.

3. Effect of the Solar Radiation Pressure

It would not be necessary at all to consider the influence of the light pressure if the satellite moving in orbit were to travel all the time in Sun rays. The force exerted by the light pressure would then be constant and could produce only secular variations in the orbit. Yet since mostly a part of the orbit lies in the area of the Earth's shadow, that factor ceases to act, and the quantity of the additional acceleration becomes dependent upon the position of satellite in orbit. Let us try to evaluate the quantities of the short-period perturbations produced by this situation. For this purpose, we shall use the method applied by Wyatt [1963] and Poliakhova [1963] , where a comparision of the effect of the solar radiation pressure and the atmospheric effect is being made.

The acceleration caused by the light pressure is expressed by the formula:

$$F_L = p\ .\ \frac{S}{m} \qquad /4.7/$$

whereby:

p - denotes the solar light pressure at the distance of the astronomical unit, exerted on a perfectly reflecting body,

S - the area of the acting cross-section of satellite,

m - the mass.

The acceleration produced by the atmospheric drag can be expressed as:

$$F_A = -\frac{1}{2}\ Cx \cdot \varrho \cdot v^2 \cdot \frac{S}{m} \qquad /4.8/$$

where:

Cx - aerodynamic coefficient,

ϱ - atmospheric density,

v - velocity of satellite.

The ratio of quantities /4.7/ and /4.8/ will be independent of the characteristics of satellite

$$\left|\frac{F_L}{F_A}\right| = + \frac{2\ p}{Cx\,\varrho\, v^2} \qquad /4.9/$$

Substituting now:

$$p = 0{,}9 \cdot 10^{-4}\ g \cdot cm^{-1}\ sec^{-2}$$
$$Cx = 2$$
$$\varrho = 67.10^{-17}\ g \cdot cm^{-3}$$
$$v = 7{,}3 \cdot 10^5\ cm \cdot sec^{-1}$$

$$\left|\frac{F_L}{F_A}\right| = + \frac{1{,}8 \cdot 10^{-4}}{714.0 \cdot 10^{-7}} = + 2.5$$

Thus, the effect of the radiation pressure at an altitude of 1000 km is some 2.5 times stronger than the atmospheric effect. Yet in Sec. 4.1 we have demonstrated that in the region of the maximal density the atmospheric effect may give perturbations of the order of a few centimeters. So, even a tenfold stronger effect will give disturbances smaller than one meter, this being below the limit of accuracy adopted by us.

As a matter of fact, the authors of the afore-mentioned studies as well as others, like Kozai [1963b] , for example, do not deal with the question of short-period perturbations, considering only secular terms which also depend on whether or not the satellite passes through the shadow of the Earth.

4. Other Possible Sources of Disturbance

We are going but mention here some other factors acting upon the motion of satellite to a markedly lesser degree than the three preceding effects. Those are:

- the effect of the Earth's magnetic field,
- the effect of the electrostatic field existing in the ionosphere,
- the effect of the radiation reflected from the Earth,
- the collisions with micrometeorites,
- the relativity effect.

This list could be complemented, according to the law of interdependence of phenomena in the Nature. Yet, all those factors produce but minimal effects, and their influence on the results of any observations is hardly perceptible.

Chapter V

APPLICATION OF THE RADIUS-VECTOR THEORY

This Chapter presents some possible applications of the theory of the artificial satellite radius-vector. They are given successively, starting with the most particular solutions imposing the fulfillment of certain specified orbital and observational conditions and ending with the quite general procedures allowing to utilize arbitrary orbits /selected, evidently, within the framework of restrictions assumed by the basic principles of the theory/ as well as observations carried out in the standard way. This sequence is also in accordance with the chronology of formation of the particular concepts.

The first one was the project of observation of the satellite ALOUETTE, presented during the Conference of Observers of Artificial Satellites in Moscow [Zieliński 1963] . We are going to discuss it hereinafter as the project of observations of a circum-polar satellite for the determination of the radius of the Earth's parallel. The other concepts have sprung up as the particular problems had arisen - such as the tetrahedron method in triangulation - and as the work on the subject-matter was progressing. There exist probably also some other possibilities of utilizing the radius-vector of the artificial satellite, susceptible to lead to interesting solutions.

1. Determination of the Earth's Parallel Radius

The objective of our program is to determine the Earth's parallel radii of the observing site for the investigation of the shape and the dimensions of the Earth. If we observe the satellite S from the point P twice during two consecutive passages through the great circle of the topocentric equator, we obtain a situation as in Fig.7: the arc represents the parallel of the observing site, or more precisely, the fragment of circle circumscribed by the parallel radius of the observing site. After a full revolution, the satellite will be almost at the same position as before /except for the translatory motion of the Earth/. During this time, the Earth will perform a revolution at an angle of σ equal to the draconic period P, this angle to be defined from the observations. The angles OP_1S and SP_2O can be determined from observations too. Assuming that we have the quantity of f, we may without difficulty solve the tetragon OP_1SP_2 and calculate the value of R /after having introduced the correction for the displacement of the point S, produced by the motion of the orbital nodes/.

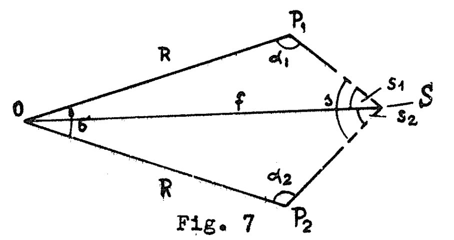


Fig. 7

From $\Delta\, OP_1S$

$$\frac{f}{\sin \alpha_1} = \frac{R}{\sin s_1}$$

From $\Delta\, OP_2S$ /5.1/

$$\frac{f}{\sin \alpha_2} = \frac{R}{\sin s_2}$$

$$\frac{\sin \alpha_2}{\sin \alpha_1} = \frac{\sin s_2}{\sin s_1}$$

yet as

$$s_1 + s_2 = s$$

thence

$$\frac{\sin \alpha_2}{\sin \alpha_1} = \frac{\sin/s - s_1/}{\sin s_1} = \sin s \cdot \mathrm{ctg}\, s_1 - \cos s$$

wherefrom

$$\mathrm{ctg}\, s_1 = \frac{\sin \alpha_2}{\sin \alpha_1 \cdot \sin s} + \mathrm{ctg}\, s \qquad /5.2/$$

Having s_1, we compute

$$R = \frac{f \cdot \sin s_1}{\sin \alpha_1} = \frac{f \cdot \sin s_2}{\sin \alpha_2} \qquad /5.3/$$

The element f is found using the formulae:

$$f = r \cdot \cos \varphi_s \qquad /5.4/$$

$$\sin \varphi_s = \sin u \cdot \sin i \qquad /5.5/$$

where: r - radius-vector
u - argument of latitude
i - inclination.

The condition of two consecutive transits may be extended to a greater number of passages during which the satellite is observed for the second time, provided that we are able to secure the required accuracy in computing the correction $\Delta \Omega$ necessary for rectifying the position of the point S during the second observation, and to safeguard the stability of the distance f, within the limits of tolerance. We assume, as we have done before, that the long-period variations are known from observations with an adequate accuracy. As regards the short-period variations, those of them which are the function of zonal harmonics of the Earth's gravity potential will be equal to zero, for the argument of latitude is the same during the first and the second observation. Only the effects of tesseral harmonics of the diurnal period are remaining. The long-period variations are minimal in the case of ALOUETTE. They are characterized by the fact that from the launching moment, that is, from the end of September 1962 till mid May 1963 the semi-major axis a did not undergo changes within the bounds of the determination accuracy: $10^{-5} \cdot R_s / R_e$ - radius of the Earth [NASA 1963] . Till the 18th May 1963, $a = 1.15892 \cdot R_o$; later on - till the end of August 1963, $a = 1.15891 \cdot R_o$.

Table 5

No of rev.	Influence of C_{22}	Influence of S_{22}	Total perturbation	$\left(\frac{\sum_1^{12}}{12}\right)$.no of rev. = secular perturbation	Total-secular= Diurnal perturbation
0	0	0	0	0	0
1	0	0	0	-1.10^{-5}	$+1^{\circ}.10^{-5}$
2	$+1^{\circ}.10^{-5}$	-1.10^{-5}	0	-2	+2
3	$+1.10^{-5}$	-2	-1	-3	+2
4	$+2.10^{-5}$	-3	-1	-4	+3
5	$+3.10^{-5}$	-6	-3	-5	+2
6	$+5.10^{-5}$	-8	-3	-6	+3
7	$+7.10^{-5}$	-11	-4	-6	+2
8	$+9.10^{-5}$	-14	-5	-7	+2
9	$+12.10^{-5}$	-18	-6	-8	+2
10	$+15.10^{-5}$	-22	-7	-9	+2
11	$+18.10^{-5}$	-26	-8	-10	+2
12	$+21.10^{-5}$	-32	-11	-11	0

On this basis and in accord with the considerations of the Chapter VII, we are arriving at the conclusions that during the interval of a dozen or so of revolutions it is not necessary to take the atmospheric drag into account.

As to the influence of the nodal motion, we know that it is little for satellites with a great inclination of the orbit. In the case of ALOUETTE, it amounts to approximately $-0^{\circ}.985/1^{d}$.

As it follows from the computations of Kotchina, corrected in such a way that the values of C_{22} and S_{22} be compatible with those adopted in the present work, the diurnal perturbations of the node of a polar satellite are: /Tab.5/.

The picture of diurnal perturbations shows a deviation from the mean variation proportional to time, by a magnitude of the order of $3^\circ \cdot 10^{-5}$, or approximately 0.″1. This corresponds to the observation accuracy 0.″6; hence the conclusion that for the computation of the correction for the alteration in the node, the application of the secular variation will do.

The influence of the inclination will be imperceptible because of the stability of that element and because of utilizing here the sine which undergoes little changes in the vicinity of 90°.

As seen from the formulae /5.4/ and /5.5/, an important role is played by the argument of latitude u:

$$d\varphi_s = \frac{\sin i \cos u \, d u}{\cos \varphi_s} \qquad /5.6/$$

$$d f = - r \cdot \sin\varphi_s \; d\varphi_s = - r \, tg \varphi_s \, \sin i \cdot \cos u \, du \qquad /5.7/$$

Substituting $i = 90^\circ$, $r = 7378$ km, and assuming that $df \leqslant 15$ m, we shall obtain the following performance of the accuracy required in defining u:

Table 6

u	|du|	dt
0	∞	
1°	28″	140 ms
10°	2.″8	14
20°	1.″5	8
30°	1.″0	5
40°	0.″8	4
50°	0.″7	3
60°	0.″6	3
70°	0.″5	2
80°	0.″5	2
90°	0.″5	2

Two milliseconds may be regarded as the upper limit of accuracy to be achieved in defining the moment of observation and in determining T_{Ω}. For observational reasons - as shown in the text to follow - it will be possible to make observations at geographical latitudes < 60°, this corresponding to $u < 48^\circ$ for an orbit of 1000 km over the Earth. So we may accept that the maximal accuracy to be achieved in the determination of u is ≈ 0.″7, this corresponding to $0.^s003(t - T_{\Omega})$. Only in the immediate neighbourhood of the equator those requirements are losing their importance.

Let us consider now the mode of observation and the influence of observational errors.

The observation should furnish - as computational elements - the angles α_1, α_2, and σ in the plane of the parallel. The angle $/180 - \alpha/$ will be the difference between the topocentric right ascension of the satellite at the moment of its transit through the equator on the celestial sphere and the local sidereal time corresponding to this moment. The angle σ will be simply the difference between the moments of both passages through the equator on the celestial sphere in sidereal time. Thus the observation will consist of the time recording and the measurement of the right ascension of the satellite when its declination is equal to 0.

Let differentiate the formulae /5.2/ and /5.3/ with regard to α_1 and α_2, considering that $d\alpha_1 = d\alpha_2 = d\alpha$:

$$\frac{- d s_1}{\sin^2 s_1} = \frac{\sin\alpha_1 \cdot \cos\alpha_2 - \sin\alpha_2 \cos\alpha_1}{\sin^2\alpha_1 \, \sin s} \, d\alpha$$

$$d s_1 = - \frac{\sin(\alpha_1 - \alpha_2) \sin^2 s_1}{\sin^2\alpha_1 \, \sin s} \cdot d\alpha \qquad /5.9/$$

$$dR = \frac{f \left(\sin\alpha_1 \cdot \cos s_1 d s_1 - \sin s_1 \cos\alpha_1 \, d\alpha\right)}{\sin^2\alpha_1} \qquad /5.10/$$

$$dR = \frac{-f \left[\sin(\alpha_1 - \alpha_2) \sin^2 s_1 + \sin s_1 \cos\alpha_1 \sin^2\alpha_1 \sin s\right] d\alpha}{\sin^4\alpha_1 \cdot \sin s} \qquad /5.11/$$

Adopting, for simplification, the symmetric observation $\alpha_1 = \alpha_2 = \alpha$, we obtain:

$$|dR| = \left| f \cdot \frac{\sin s_1 \cos \alpha_1}{\sin^2 \alpha_1} \cdot d\alpha \right| \qquad /5.12/$$

For such an observation, with an orbit of $H = 1000$ km and the geographical latitude of the observing site $\varphi \cong 52^\circ$, we have:

$$\alpha_1 = \alpha_2 \cong 138^\circ$$
$$S_1 \cong 29^\circ$$
$$f \cong 5500 \text{ km}$$

$d\alpha$ will stand for 2".

Therefore:

$$|dR| = 20 \text{ m}$$

The assumption that the measurement error of angles α is equal to $\pm$ 2" seems somewhat optimistic. In addition to the very measurement accuracy of the right ascension, there is also the question of the determination accuracy of the local sidereal time, which is conditioned on the accuracy known to be for the g e o c e n t r i c geographical longitude of the observing site. For the accuracy of this element to be below 2", it means, in order to avoid the error of deflection of the vertical, we must have geodetic coordinate data in one of the newer systems /for instance: Hayford, Kroscwski/, in which the error of their orientation is supposed not to exceed 2". Those coordinates need, moreover, to be reduced to the position of the instantaneous pole. The case where only astronomically determined coordinates are known will be discussed separately.

The accuracy achieved in the measurement of the angle δ is considerably higher. If we measure the time with an accuracy of $\Delta t = \pm 0^s.002$, then $\Delta \delta = \pm 0''.03$; so we are not going to analyse here this influence.

Assuming that $mf = 15$ m and $m\alpha = 2''$, we obtain

$$m_R = 25 \text{ m.}$$

As said before, this estimate may be regarded as being too optimistic, but our observation will prove to be of avail even when m_R is much greater. In this connection we may recall that the discrepancies between the dimensions of different ellipsoids are still of the order of 100 + 200 m; the same is with the position of the center of the Earth´s mass. And so, if a single observation is able to supply an information of the same class as the complicated and time-consuming measurement of large areas made with the help of geodetic methods, the accuracy achieved in this case may be considered interesting.

Let us now dwell upon the question of the visibility of the satellite. It is not very often that we have the possibility of observing two consecutive transits; yet, as said before, this fact reveals no greater significance because of the stability of the selected orbit. Instead, it is more important that the satellite be observed by many other stations in order to be able to best determine the necessary orbital elements. There are, unfortunately, periods during which the visibility conditions allow only for a very short arc of the satellite to be observed. This occurs when the line of the orbital nodes is near the Earth-to-Sun direction. This effect is stronger with lower orbits and feebler with higher orbits. Therefore it is more advantageous to choose higher orbits. Next to ALOUETTE - MIDAS 3 and MIDAS 4 with orbits at an altitude of approximately 3500 km, or other satellite, may be included into the observation program.

The visibility conditions define, also the range of geographical latitudes at which the observations of this type can be carried out. Since the observation has to be performed when $\delta = 0$, and on a fairly large arc of the equator, it is restricted by the maximal zenithal distance at which an observation is possible. The corresponding formulae for the computation are given by Cichowicz [1963] . It follows thereof that for $z_{max} = 85^\circ$ and in the range of $\pm 30^\circ$ of the hour angle, φ must be smaller than 60°.

2. Computation of the Triangulation by the Tetrahedron Method

This method has been described in detail and results have been presented in previous works by the author, [Zieliński 1965], [Pachelski, Zieliński 1966]. In this paper only the general conception will be recalled with some final conclusions.

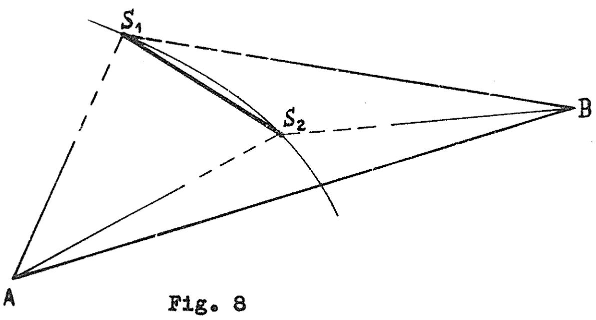


Fig. 8

If we observe the satellite from the point A at two positions S_1 and S_2 corresponding to two moments T_1 and T_2, we obtain the AS_1, AS_2, BS_1, BS_2 directions in the system of absolute orientation, it means, in such one whose Z-axis is covered by the rotation axis of the Earth, and the XY-plane - by the equatorial plane. If, knowing the orbital elements, we are going to calculate the length of the segment of the straight line S_1S_2, we shall be able to solve readily the tetrahedron AB S_1S_2, and to find the distance and the direction AB. Choosing arbitrarily the coordinates of the point A, we can compute the coordinates of the point A and, further on, in the same way the coordinates of all points of the network. The points of the computed network submitted afterwards to an adjustment, will obtain their coordinates in the absolutely oriented system with the length unit independent of the distance measurements performed on the Earth's surface and, hence, disencumbered of measurement errors of bases, errors of reduction from the geoid to the ellipsoid, a.s.o. Also another source of errors disappears, connected with the transposition of the length from the measured side to the unmeasured sides, because the number of "bases" exceeds markedly the number of sides.

Thanks to the participation of Poland in the International Observation Program of the Satellite ECHO 1, in 1963, a possibility was offered for the test of this concept by using the observational data collected. For this purpose, analytical formulae have been derived for the length of the base segment $S_1S_2 = \ell$ in the function of orbital elements, and a theoretical accuracy analysis has been made; the main idea of the experiment was to demonstrate the validity of this analysis. If it proved to be valid for ECHO 1, it ought to be good also for other satellites which fulfilling all the orbital and aerodynamic requirements, will secure much more accurate results than the balloon ECHO. The computations gave the agreement of results as predicted by the theoretical analysis or even better. According to this analysis the accuracy of determination of the segment ℓ in the space from the orbital elements of satellite Echo 1 is of the order $\pm$ 170-190 cm, for the case geometrically advantageous. From observations we have:

Table 7

	ℓ_{cal}	ℓ_{geod}	m_ℓ	m_o
Riga-Uzhgorod	932.096 km	931.932 km	$\pm$ 94 m	$\pm$ 250 m
Riga-Nikolaev	1232.409 km	1232.336 km	$\pm$ 57 m	$\pm$ 127 m

ℓ_{cal} - calculated from orbital elements
ℓ_{geod} - calculated from geodetic coordinates
m_ℓ - mean error of the result
m_o - mean error with unit weight

m_ℓ and m_o bear also upon some observational errors not discussed in theory. If compared to the results derived by other orbital methods, e.g. Veis [1964], which being based on tens of thousands of observations had given mean-square errors within the limits 7-30 m, our results may be regarded as encouraging.

3. Connection of the Triangulation Network with the Center of the Earth's Mass

The method described in Sect.5.2, enables to determine the coordinates of all points in a system with strictly defined directions but with an arbitrarily chosen origin.

Having evaluated the coordinates of points A and B, we can also compute the coordinates of the two remaining vertexes of the tetrahedron – S_1 and S_2. Since we know the length of the radius-vector at this point, we may put down:

$$\left.\begin{aligned}(X_{S1} - X_0)^2 + (Y_{S1} - Y_0)^2 + (Z_{S1} - Z_0)^2 &= r_1^2 \\ (X_{S2} - X_0)^2 + (Y_{S2} - Y_0)^2 + (Z_{S2} - Z_0)^2 &= r_2^2\end{aligned}\right\} \qquad /5.13/$$

where Xo, Yo, Zo are the coordinates of the mass center.

As many such equation systems can be written as many tetrahedrons have been formed in the network, this allowing for the coordinates Xo, Yo, and Zo in the adopted system to be found. After having made a parallel shift, we obtain a coordinate system with the origin at the center of the Earth's mass.

During the last years, several projects of the world triangulation network have been presented. One of the most elegant and – at the same time – one of the most efficient projects is the Zhongolovich project [1965]. This project posssesses, however, some limitations consciously imposed by the author himself who considers certain difficulties to be solvable only in the future. Among those difficulties is the question of conveying the spatial network a linear scale. Zhongolovich suggests to apply here laser-beacon for measuring one of the segments: satellite-observing station. Stating expressively that this will be a task of the future, he draws the attention to a whole series of technical problems bound with it.

The application of the method described in the present study eliminates this predicament and enables to acquire a great number of redundant data under the form of measured distances.

Another problem that the author of the world triangulation network has left irresolved is the question of relating the network to the center of the Earth's mass, although he states in the introduction that this is the very ultimate objective of any studies of this kind.

The method applying the radius-vector allows also for that part of the project to be complemented.

4. Determination of the Parallel Radius by Using the Tetrahedron Method

In Sect.5.1 the method of determining the parallel radius of the Earth has been discussed, in which an accurate knowledge of the geocentric distance of the observing site was being assumed. Let now give a thought to the question how we have to proceed if this condition is not satisfied.

A twofold aiming at the point S which – in reality – is moving and is considered motionless only owing to certain

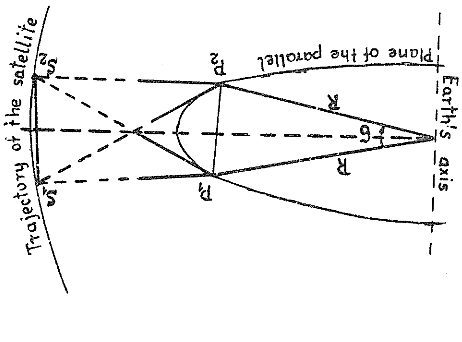


Fig. 9

corrections introduced into the calculus, may be called q u a s i - s y n c h r o n o u s. The same quasi-synchronism can be applied to the observation of other selected points, for example, points remote in time from the point S, by an interval of $\pm \Delta t$. Thus, we are able to construct a tetrahedron analogous to the tetrahedron used in triangulation, whereby the place of the previous two points A and B will be taken by a single point P which - during the time separating two observations - would have changed its position in space. In the issue, we shall obtain the distance of P_1P_2 permitting to determine R immediately thereof:

$$R = \frac{P_1P_2}{2} \cdot \operatorname{cosec} \delta/2 \qquad /5.14/$$

Now we may venture to define an element which is perhaps even more interesting than R, namely - the parallel component of the deflection of the vertical. For this purpose, the formula /5.11/ from the Sect.5.1 can be utilized. We shall rewrite it in the form:

$$\eta = \frac{-\Delta R \cdot \sin^4 \alpha_1 \sin S}{f\left[\sin(\alpha_1 - \alpha_2)\sin^2 S_1 + \sin S_1 \cos \alpha_1 \sin^2 \alpha_1 \sin S\right]} \qquad /5.15/$$

where: η - component of the vertical deflection

ΔR - difference between the radius determined in the manner described in 5.1 $/R_I/$ and the radius determined in the way described here $/R_{II}/$.

$$\Delta R = R_{II} - R_I \qquad /5.16/$$

The quasi-synchronization of points S_1 and S_2 may be achieved with the same accuracy as of the point S. Only minute variations of elements come into play here, occurring during the time of Δt 1-2 min; they can be easily taken care of.

5. Synchronous Observations Not Forming Tetrahedrons

The last of two methods we are going to present here, are the most complicated as a matter of analysis, but they offer the possibility of utilizing the existing observational material as well as any other observations to be made in the future with the help of the standard procedures.

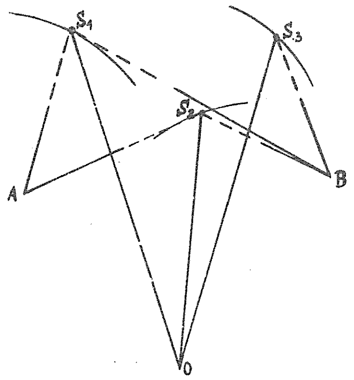


Fig. 10

Let suppose that synchronous observations of a satellite or of satellites being at points S_1, S_2, S_3 are to be carried out from two points A and B. Each of those observations will permit for the following equation system to be formulated:

$$\left.\begin{aligned}
&\frac{y_{si} - y_A}{x_{si} - x_A} = tg\, t_{Ai} \qquad\qquad \frac{y_{si} - y_B}{x_{si} - x_B} = tg\, t_{Bi} \\
&\frac{z_{si} - z_A}{\sqrt{(x_{si} - x_A)^2 + (y_{si} - y_A)^2}} = tg\, \delta_{Ai} \qquad \frac{z_{si} - z_B}{\sqrt{(x_{si} - x_B)^2 + (y_{si} - y_B)^2}} = tg\, \delta_{Bi} \\
&x_{si}^2 + y_{si}^2 + z_{si}^2 = r_i^2
\end{aligned}\right\} \qquad /5.17/$$

where: x_{si}, y_{si}, z_{si} - are the satellite coordinates at the moment of the i-th observation,
$x_A, y_A, z_A, x_B, y_B, z_B$ - are the coordinates of the stations,
t_{Ai}, t_{Bi} - are the hour angles referred to Greenwich, observed from stations A and B
δ_{Ai}, δ_{Bi} - are the declinations observed from A and B.

Three observations will give a system of 15 equations with 15 unknowns. The solution of a system in which three equations are of the second degree is certainly not an easy matter but after all, not hopeless. As the coordinates of points A and B are always known with a fairly good approximation and as the coordinates of points S_1, S_2 and S_3 can be evaluated to a slightly worse approximation, the numerical method of iteration may be applied without fear of the process to prove discrepant.

Apart from the accuracy of observations and the accuracy of radii-vectors, the accuracy of such a determination will depend on the geometrical configuration. The most favorable conditions will prevail when the directions at points A and B form approximately right angles with each other or, in other words, when the points S_1, S_2 and S_3 are largely spaced. It follows thereof that the distance between the stations should not exceed the altitude of the satellite above the Earth, otherwise, it would be hardly possible to attain a favorable configuration.

6. Nonsynchronous Observations

Let us suppose that at the station A the observations are made in such a way that during one flight the satellite is being observed at two or more places on the sky, at intervals of several minutes. This allows to evaluate the length of the segments l in space, analogous to the base segments appearing in tetrahedrons. We are going to consider three of such observations. Each of them permits for the following equation to be set down:

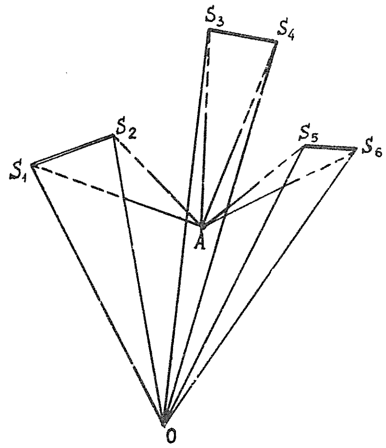


Fig. 11

$$\left.\begin{aligned}
&\frac{y_1 - y}{x_1 - x} = \mathrm{tg}\, t_1 \qquad \frac{y_2 - y}{x_2 - x} = \mathrm{tg}\, t_2 \\
&\frac{z_1 - z}{\sqrt{(x_1 - x)^2 + (y_1 - y)^2}} = \mathrm{tg}\,\delta_1 \qquad \frac{z_2 - z}{\sqrt{(x_2 - x)^2 + (y_2 - y)^2}} = \mathrm{tg}\,\delta_2 \\
&(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = l_{12}^2 \\
&x_1^2 + y_1^2 + z_1^2 = r_1^2 \\
&x_2^2 + y_2^2 + z_2^2 = r_2^2
\end{aligned}\right\} \qquad /5.18/$$

where: $x_1, y_1, z_1, x_2, y_2, z_2$ - are the coordinates of the points S_1 and S_2,
x, y, z - are the coordinates of the observing site,
$t_1, t_2, \delta_1, \delta_2$ - are the observed hour angles and declinations respectively.

The three observations give a system of 21 equations with 21 unknowns.

The matter of solving this system stands like in the preceding case. Owing to the fact that we know the approximate coordinates of all points, we are in a position to solve this system fully uniquely with the aid of the iteration method, although some more real solutions exist here.

As it always is, the accuracy depends strongly upon the configuration. So, for the point A to be properly determined a good location of observed points S_i is needed.

Chapter VI

FINAL CONCLUSIONS

The considerations presented in this study and the outcomes of computations made in connection with the work, lead to the following conclusions:

1^o If we have at our disposal a set of observations carried put with an accuracy to $\pm 2''$ and $\pm 0^s.002$, well distributed in space and time, it is possible to compute the orbital elements, especially the semi-major axis and the eccentricity, with an accuracy better than 10^{-6}. Applying therewith an appropriate observation program, we can disburden completely the semi-major axis and partially the eccentricity of position errors of observing stations /Chapter II/.

2^o The present state of knowledge on the Earth's gravity field permits to compute the radius-vector accurate to 10^{-5} - if the satellite is travelling on an orbit of H_p equal to 1000 km and i being equal to 80^o. It is the lowest orbit within the adopted limitations H = 1000 ÷ 3000 km - having been purposely chosen for our calculations as the least favorable. With higher orbits the effect of harmonics of higher orders is rapidly dropping because of the high powers r appearing in the denominator /Chapter III/.

3^o Perturbation influences of atmospheric provenance produced by the attraction of other celestial bodies or generated by other causes - are small and need not be taken into consideration /Chapter IV/.

Let reflect on the question in what sense those results offer possibilities of utilizing the theory of radius-vector. Although the accuracy to 10^{-5} may be not very attractive for problems being resolved on the Earth's surface such for example, as the connection of continents, but becomes rather interesting with the problem of referring surface points to the center of the Earth's mass. The studies existing so far in this domain did not pass the mentioned accuracy limit, neither. However, the investigations on the Earth's gravity are making a rapid progress. The accuracy known for the C_{22} and S_{22} harmonics responsible for the largest perturbations, is continually improving and will, no doubt, in the nearest future attain the same order of accuracy as the zonal harmonics. This gives foundation for hope that it will be possible to achieve also with regard to this part of the theory an accuracy of the order of at least 2.10^{-6} ÷ 3.10^{-6}. Still, it seems necessary to remark that it may be hardly possible to improve here the accuracy by means of augmenting the number of observations because the errors, having their source in the accuracy of the theory, would not be of an accidental nature.

The author fully realizes that the present study does not exhaust the subject matter. It would be, primarily, very desirable to undertake the integration for other types of orbits with different inclinations and altitudes above the Earth's surface. An examination of the influence of tesseral harmonics of the third and the fourth order would also be of avail. Such calculations inexecutable without the aid of electronic computers are connected with rather large expenditures. It is to be hoped, however, that this investigation work will be continued.

It is the most pleasant duty of the author to express his gratitude to all persons who have not spared advice or support during his work on the above problem:

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- to the personnel of the Establishment for Numerical Computations of the Warsaw University, where the calculation operations had been carried out.

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ПРИМЕНЕНИЕ РАДИУСА-ВЕКТОРА ИСКУССТВЕННОГО СПУТНИКА ЗЕМЛИ В ГЕОДЕЗИИ, КАК МЕРЫ ДОЛГОТЫ

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Используя третий закон Кеплера, можно определить полуось орбиты спутника на основании заобсервированного периода его обращения. Если известен также эксцентриситет орбиты, то принимая, что он близок нулю, можно вычислить радиус-вектор спутника для произвольного момента, даже в случае, если остальные элементы известны лишь приблизительно. Настоящая работа посвящена главным образом исследованию, с какой точностью является возможным определение радиуса-вектора. При предпосылках, что эксцентриситет орбиты близок нулю, что высота перигея 1000-3000 клм. а соотношение массы /поверхности спутника велико, оказывается возможным вычисление радиуса-вектора с точностью 10^{-5}. Основным препятствием в достижении большей точности является недостаточность сведений о коэффициентах тессеральных гармоник земного гравитационного поля. Прочие причины возмущений, как то: ошибка коэффициентов зональных гармоник, сопротивление атмосферы, давление света, притяжение Солнца и Луны - вызывают гораздо меньшие пертурбации. Приведен также способ наблюдений и вычисления элементов орбиты, элиминирующий влияние ошибок координат мест наблюдений. В последнем разделе работы приведено несколько способов использования известного радиуса-вектора для определения координат точек на поверхности Земли в геоцентрической системе, ориентированной соответственно с направлением оси вращения и плотности экватора.