# TORSION OF THE EARTH'S ANOMALOUS GRAVITATIONAL FIELD RESULTING FROM THE FINITE SPEED OF THE GRAVITATIONAL INTERACTION

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The external gravitational field of the Earth is not axially symmetric but is irregular, described by series of the spherical harmonic functions. In the local inertial coordinate system the field is rotating following the rotation of the Earth. In the Newtonian interpretation the field is rotating stiffly with the Earth but according to the Einstein's General Relativity interpretation the propagation speed of the gravitation is finite and the gravitational anomalies are propagated in space with the speed of light. Consequently the anomalous field at the altitude $h$ should be twisted by angle $\alpha = h\,\omega/c_g$ comparing to the ground level ($\omega$ - the speed of the Earth's rotation, $c_g$ – speed of the gravitational signal). This effect is difficult to measure because of the decreasing of anomaly values with the distance. However, with the modern techniques like GNSS positioning and gradiometry the torsion could be measured.

*Key words*: Earth gravity field, Earth rotation, speed of the gravitation

## 1. Introduction

In classical formulation the external gravitational potential of the body with the mass $M$ is:

$$V(r,\phi,\lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n} P_{nm}(\sin\phi)\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right] \qquad (1)$$

In this expression $V$ is a function of coordinates $r, \phi, \lambda$ of the point external to the sphere $\boldsymbol{E}$ of the radius $a$, assuming there are no masses outside of this sphere. Coefficients $C_{nm}$ , $S_{nm}$ are dimensionless quantities, called gravitational multipole moments or Stokes coefficients and they depend on the distribution of masses inside of the sphere $\boldsymbol{E}$. Derivation of this expression is shown in [Heiskanen, Moritz, 1967, p.59].
The above formulation is done for the static body being in rest in the inertial reference frame. In the case of the Earth the rotation has to be accounted:

$$W = V + F \qquad (2)$$

where $F$ – potential of the centrifugal force, but (1) holds when we discuss the gravitation only.
If we put

$$V = const \qquad (3)$$

we get the equipotential surface, the special one possibly closest to the surface of the global ocean is called *geoid*. But near the Earth, where $r \approx a$, the shape of the equipotential surfaces $V_r$ differs from the sphere and even from the ellipsoid, it is irregular. The irregularities, called anomalies, described by the summation term in (1), diminish with the growing distance, nevertheless they exist.
The derivatives of $V$ with respect to coordinates give gravitational accelerations while the tensor of second derivatives is called the gravitational gradient $T$. For $T_{rr}$ – the gradient on the radial direction – we have the expression:

$$T_{rr} = \frac{1}{r^2}\sum_{n=2}^{N}(n+1)(n+2)\left(\frac{a}{r}\right)^{n+1}\frac{GM}{a}\sum_{m=0}^{n}\left(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right)\bar{P}_{nm}(\sin\varphi) \qquad (4)$$

The Eötvös Unit: 1 EU = $10^{-9}$ m $s^{-2}$ /m is used for quantifying the gradient.
Now let us remind that the Earth is rotating. The rotation is practically modeled in the quasi-inertial reference frame, defined as a frame with axis oriented to the very distant radiosources but with the origin in the Earth's center of mass. Let call it the Earth's centered inertial frame. According to the Newtonian physics the whole set of equipotential surfaces $V_r$ rotates uniformly with the Earth itself, independently

from the growing *r*. In such a frame we use the angle *θ(t)* being the function of the sidereal time *t* instead of the coordinate *λ*. Substituting it to (1) we get the expression for *V* in the local inertial frame.
During the last years we had a number of excellent results of the modeling of the Earth gravitational field thanks mainly to the two space missions – GRACE and GOCE [Tapley et al., 2012],[Rummel, 2011].

## 2. GR approach to the description of the Earth Gravitational Field

At the beginning of the era of the satellite geodesy the General Relativity has not been considered essential for practical determinations because of the marginal influence of the relativistic effects. However, the situation changed with the introduction of the GPS constellation. This system is entirely based on time measurements – from the determination of time the geometric parameters are determined. For the definitions of the time scale or simultaneity the General Relativity is not marginal any more.
Works of Ashby [Ashby, 2003], [Ashby, 2004], draw the fascinating picture of the most modern satellite system seen in the relativistic environment. The definitions of and transformation between the Earth Centered Inertial Frame and Earth Fixed Reference Frame give very solid foundation for further discussion of the working elements of the system. Problems of timing are most thoroughly discussed.
The phenomenon of the Earth rotation in GR is thoroughly studied in the context of the Lense-Thirring Effect. The significant space mission Gravity Probe B has been accomplished [Everitt at al.,2011] and orbital analysis of the satellites Lageos and LARES provides the new proofs of it [Ciufolini at al.,2012].
In this report we are trying to investigate the particular consequences of the important assumption of the general relativity theory – the finite speed of the propagation of gravitation.
In the Newtonian physics the law of universal gravitation is not time dependent, what means that the force between two objects is transmitted instantaneously, no matter how far the objects are. According to Einstein special relativity principle no signal can travel in space faster than the speed of light, including gravitational interaction. The assumption that the speed of the propagation of the gravitational interaction is equal to the speed of light in vacuum is a fundamental tenet of the general relativity theory.
For the planet Earth we can follow the reasoning of Ashby, describing the GPS system. This system is designated for positioning and navigation on or near the Earth's surface, that is in the Earth Fixed Reference Frame. But in order to work properly it must be placed in the inertial system. Let me quote Ashby [Ashby,2004, p.12] literally: "Einstein's Principle of Equivalence allows one to discuss frames of reference which are freely falling in the gravitational fields of external bodies. Sufficiently near the origin of such a freely falling frame, the laws of physics are the same as they are in inertial frame; in particular electromagnetic waves propagate with uniform speed *c* in all directions. Such freely falling frames are called *locally inertial* frames. For the GPS, it is very useful to introduce such a frame that is nonrotating, with its origin fixed at Earth's center, and which falls freely along in the Earth in the gravitational fields of the other solar system bodies. This is called an Earth Centered Inertial (ECI) Frame". In this frame we have to describe the gravitational field.
We are entitled to assume that the gravitational interaction, later called the *gravitational signal* behaves in the same way as electromagnetic signals with the same speed in the inertial frame ECI.
Now let consider the path of the photon emitted radially from the light source on the rotation axis [Grøn, 2009]. In ECI frame photon path is a radial straight line. But in Earth Fixed Rotating (EFR) frame rotating with angular velocity *ω* the radial coordinates *r* and *λ* will be

$$r = t \cdot c, \qquad \lambda = \omega \cdot t \tag{5}$$

In EFR frame the photon path equation will be

$$\lambda = \frac{\omega \cdot r}{c} \tag{6}$$

which is an equation of Archimedean spiral.
If we take the tangent to this spiral at the point at distance *r,* the difference between the tangent and the radial directions gives us the effect of the rotation on the observed direction of the distant source. This effect – measured on the Earth surface - is called in astrometry the *diurnal astronomical aberration.*

By analogy, the gravitational signal emitted from the point mass located in the origin of IF will travel along the similar path. Let us denote it by *α*:

$$\alpha = \frac{\omega \cdot r}{c_g} = \frac{\vec{v}_r}{c_g} \operatorname{cosec} 1'' \tag{7}$$

where

$\vec{V}_r$ - tangential velocity proportional to rotation at the distance *r*,

$c_g$ – speed of the gravitational signal.

If $c_g \neq \infty$ and the body is in motion the potential in the external point will be retarded by the time needed for the transmission of the signal from this body to P. It is called Lienard - Wiechert potential [Landau,Lifshitz,1975].

Applying it to the rotating Earth we have the mass of the Earth distributed within the Earth's body, moving around its axis with respect to the Earth Centered Inertial Frame. The motion with respect to the external point will be measured by the rotation velocity $\omega$ and distance $r$. If for $r$ we take the distance run by the gravitational signal from the origin to the external point then the increment of the direction will be equal to $\alpha$ according to (7). It will be in the equatorial plane of the ECIF therefore the spherical coordinates are $r, \varphi, \theta+\alpha$.

Following the same path of the derivation as for (1) we get the expression

$$V = \frac{GM}{r}\left[1+\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^n \left(C_{nm}\cos m(\lambda+\alpha)+S_{nm}\sin m(\lambda+\alpha)\cdot P_{nm}(\sin\varphi)\right]\tag{8}$$

for the retarded potential of the rotating Earth in the EFR frame. The angular increment $\alpha$ is linearly increasing with the distance from the rotation axis. We can interpret this formula in such a way that the potential $V$ - like onion - consists of layers of equal potential, each one twisted by the angle $\alpha_r$, with respect to the $V_o$, called geoid.

In the Newtonian interpretation the field is rotating rigidly with the Earth. In the local inertial coordinate system the field is rotating following the rotation of the Earth.

In the post Newtonian interpretation the distant gravity field with its equipotential surfaces is not able to rotate with the same phase angle as the Earth.

As $\alpha$ is growing with $r$ we can analyze the field on different heights above the Earth: when the gravitational signal from the Earth reaches the height $H_1$ at the time $t_1$ and the height $H_2$ at $t_2$, the earth will rotate by the angle $\Delta\alpha$

$$\Delta\alpha = \Delta t\omega = \frac{H_2 - H_1}{c_g}\omega \tag{9}$$

where:

$\Delta t = t_2 - t_1$

ω – angular velocity of rotation.

### 3. Possibility of the measurement of the gravitational signal propagation effect.

If the above reasoning is correct, then the real gravitational field will differ from the model expressed by equation (1) based on the Newtonian physics. We will have the angular shift depending on the altitude of the equipotential layer above the geoid. The $\Delta\alpha$ can be calculated from (9). For the altitude difference 20000 km, what is the orbital height of the GPS satellites, we have $\alpha = 1''$. In terms of the angular units it is rather small value, but in terms of the distance in space it is of the order of 130 m what is not a minuscule value. The question is if it could be measured and by what means.

In geodesy, the upward continuation procedure is used to calculate the gravitation on higher levels from the model given for the low level without the loss of precision. Of course the higher we are going the smaller are anomalies in term of absolute values and the model is going to be more and more smooth. The computation of upward continuation can be performed with arbitrary precision but the measurement accuracy depends on the technology available, of course.

Among parameters of the field the practically observable in space is the gradient of the Earth gravitational acceleration. Models published from GOCE mission are in terms of the geoid undulations and gravity anomalies, but also three-dimensional gradients [Rummel,2011]. The declared sensitivity of the gradiometer is $\pm 3.10^{-3}$ EU. However, the parameter interesting for us here is the orientation of the gravity field model with respect to the coordinate $\lambda$ and change of this orientation with height H. Sensitivity of the model to the change in $\lambda$ could be approximately estimated from the analyze of the expression (4). For the height level 200 km we can conclude that the estimation of $m_\lambda \approx \pm 0''.1$ is safe.

Models given by GOCE refer to the height level ca. 250 km. What can be said about the orientation accuracy on higher levels ? New mission and experiments could be envisaged. E.g., if we select the level

3000 km, the gradiometric signal variation is of the order of 0.1 EU/ $30^0$, and we have time interval $\Delta t$ = 0.01 s and $\alpha$ = 0".15. For the measurement of such an angle, the orientation precision resulting from gradiometry should be improved approximately by two orders. If so, we could conclude that with gradiometry precision $10^{-5}$ EU our proposed experiment for the measurement of the angle $\alpha$ could be successful.
However, other techniques could be considered too. Let us imagine that in the future GPS and Galileo satellites will be equipped with accelerometers, laser ranging devices and range rate systems imitating the GRACE mission. This new generation GNSS constellation could provide the very accurate gravitational field model for the height region 20000 – 23000 km. With such data the angle $\alpha$ might be determined with sufficient accuracy.

## 4. Conclusions

If the torsion angle $\alpha$ could be measured then the velocity of the gravitation $c_g$ could be determined by (9). The rotating Earth would be used as the signal generator, and GNNS plus gradiometry as detector. It seems that it is realistic, but of course not easy. However, we have to admit, that many questions are still not answered. The question of the accuracy of the determination of $c_g$ requires further study and simulations. If successful this kind of experiment could have consequences for the fundamental physics research of gravitation. If the assumption

$$c_g = c$$

will be confirmed with sufficient accuracy then the General Relativity theory will pass another important test [B.S.Sathyaprakash, B.Schutz,2009].

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