Chaos and Multifractals in the Solar System Plasma

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Abstract

We argue that dynamical behaviour of space plasmas can often be approximately described by low-dimensional chaotic attractors in the inertial manifold, which is a subspace of a given system phase space. In fact, using nonlinear time-series analysis based on the method of topological embedding a chaotic strange attractor has been identified in the solar wind data (Macek, 1998) as further examined, e.g., by Macek *et al.* (2005). In particular, we have shown that the multifractal spectrum of the solar wind attractor is consistent with that for the self-similar generalized weighted Cantor set with one probability measure parameter of the chaotic attractor and one or possibly two scaling parameters describing nonuniform compression in the phase space of the system. The values of the parameters fitted demonstrate small dissipation of the complex solar wind plasma and show that some parts of the attractor in phase space are visited much more frequently than other parts.

To quantify the multifractality of space plasma turbulence, we have recently considered that generalized two-scale weighted Cantor set also in the context of solar wind intermittent turbulence (Macek and Szczepaniak, 2008). We investigate the resulting multifractal spectrum of generalized dimensions depending on parameters of the new cascade model, especially for asymmetric scaling. In particular, we show that intermittent pulses are stronger for the model with two different scaling parameters a much better agreement with the solar wind data is obtained, especially for the negative index of the generalized dimensions.

Therefore we argue that there is a need to use a two-scale cascade model. We hope that this generalized multifractal model will be a useful tool for analysis of intermittent turbulence in the Solar System plasma. We thus believe that fractal analysis of chaotic phenomena in the complex space environment could lead us to a deeper understanding of their nature, and maybe even to predict their seemingly unpredictable behaviour.

Within the complex dynamics of the solar wind's fluctuating intermittent plasma parameters, there is a detectable, hidden order described by a strange chaotic attractor that exhibits a multifractal structure.

Plan of Presentation

1. Introduction
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2. Solar Wind Data
   - Solar Wind Fluctuations
   - Space Missions (Helios, Ulysses, ACE/WIND)
3. Methods for Solar Wind Attractor
   - Generalized Dimensions
4. Methods for Turbulence Models
   - Structure Functions Scaling (Energy Transfer Rate)
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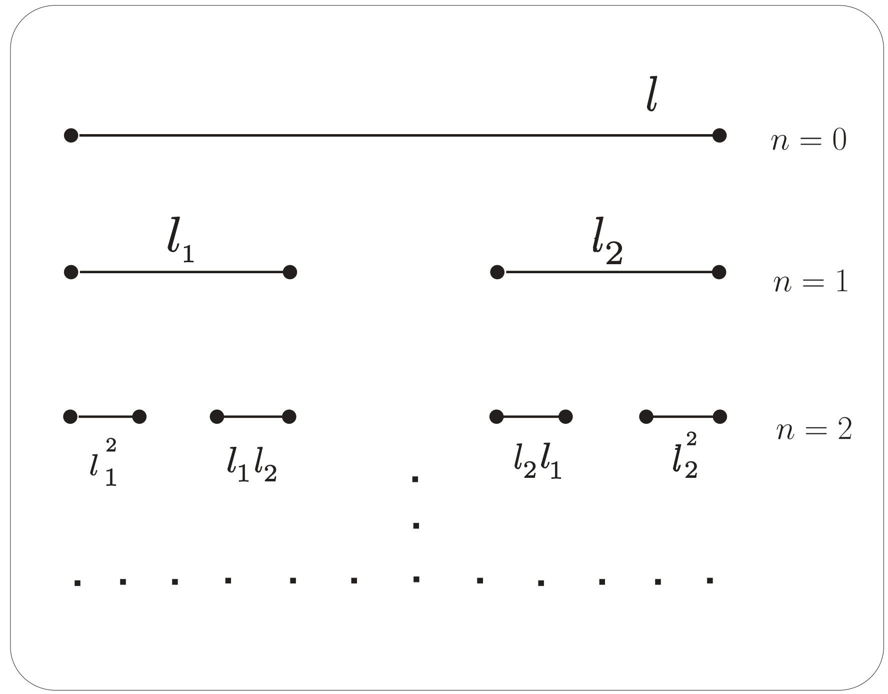
   - Generalized Two-Scale Mutifractal Model for Intermittent Turbulence
   - Comparison with the Usual p-model
6. Conclusions

Prologue

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally *self-similar* and independent of scale (fractal dimension).

A **multifractal** is a set of intertwined fractals. Self-similarity of multifractals is scale dependent (spectrum of dimensions).



Two-scale **Cantor** set.

Chaos and Attractors

CHAOS ($\chi\alpha o\varsigma$) is

- APERIODIC long-term behavior
- in a DETERMINISTIC system
- that exhibits SENSITIVITY TO INITIAL CONDITIONS.

A positive finite Lyapunov exponent (metric entropy) implies chaos.

An **ATTRACTOR** is a *closed* set A with the properties:

1. A is an INVARIANT SET:
   any trajectory $\mathbf{x}(t)$ that start in A stays in A for ALL time t.
2. A ATTRACTS AN OPEN SET OF INITIAL CONDITIONS:
   there is an open set U containing A ($\subset U$) such that if $\mathbf{x}(0) \in U$, then the distance from $\mathbf{x}(t)$ to A tends to zero as $t \to \infty$.
3. A is MINIMAL:
   there is NO proper subset of A that satisfies conditions 1 and 2.

The nature of the fluctuations in solar wind plasma parameters is still very little understood. The slow solar wind most likely originates from nonlinear processes in the solar corona. However, it appears that a certain kind of order does lie concealed within the irregular solar wind fluctuations, which can be described using methods of nonlinear time series analysis, based on fractal analysis and the theory of deterministic chaos. This involves the notions of fractal and multifractal sets, which could presumably be strange attractors in a certain state space of a given complex dynamical system. By employing the so-called false-nearest-neighbors method, we have argued that the deterministic component of solar wind plasma dynamics should be low-dimensional (Macek and Strumik, 2006). In fact, the results we have obtained using the method of topological embeddings indicate that the behavior of the solar wind can be approximately described by a low-dimensional chaotic attractor in the inertial manifold, which is a subspace of system phase space.

A direct determination of the solar wind attractor from the data is known to be a difficult problem. This chaotic strange attractor has been identified in the solar wind data by Macek (1998) as further examined by Macek and Redaelli (2000). In particular, Macek (1998) has calculated the correlation dimension of the reconstructed attractor in the solar wind and has provided tests for this measure of *complexity* including statistical surrogate data tests (Theiler *et al.*, 1992). Further, Macek and Redaelli (2000) have shown that the Kolmogorov entropy of the attractor is *positive* and finite, as it holds for a *chaotic* system.

The question of multifractality is of great importance because it allows us to look at intermittent turbulence in the solar wind (e.g., Marsch and Tu, 1997; Bruno *et al.*, 2001). Starting from Richardson's scenario of turbulence, many authors try to recover the observed scaling exponents, using some simple and more advanced fractal and multifractal models of turbulence describing distribution of the energy flux between cascading eddies at various scales. In particular, the multifractal spectrum has been investigated using Voyager (magnetic field) data in the outer heliosphere (e.g., Burlaga, 1991, 2001) and using Helios (plasma) data in the inner heliosphere (e.g., Marsch *et al.*, 1996). The multifractal scaling has also been tested using Ulysses observations (Horbury et al., 1997) and with ACE/WIND data (e.g., Hnat *et al.*, 2003, 2007; Chapman *et al.*, 2006; Kiyani *et al.*, 2007).

We have also analyzed the spectrum for the solar wind attractor. This spectrum has been found to be consistent with that for the multifractal measure of the self-similar weighted baker's map with two parameters describing uniform compression and natural invariant probability measure of the attractor of the system (Macek, 2002, 2003, 2006, 2007; Macek et al., 2005, 2006).

Recently, in order to further quantify the multifractality, we have considered the generalized weighted Cantor set with two different scales describing nonuniform compression also in the context of turbulence cascade (Macek and Szczepaniak, 2008). Even thought one can find this general Cantor set in many classical textbooks, e.g., (Falconer, 1952, Ott, 1993), it is still difficult to understand this strange attractor that exhibits multifractality in various complex real systems, also in case of intermittent turbulence.

Hence we have argued that there is, in fact, need to use a two-scale cascade model. Therefore, we investigate the resulting multifractal spectrum of the energy flux depending on two scaling parameters and one probability measure parameter, demonstrating that intermittent pulses are stronger for asymmetric scaling and a much better agreement is obtained especially for $q < 0$. We hope that this generalized new asymmetric multifractal model could shed light on the nature of turbulence and will be a useful tool for analysis of intermittent turbulence in various environments.

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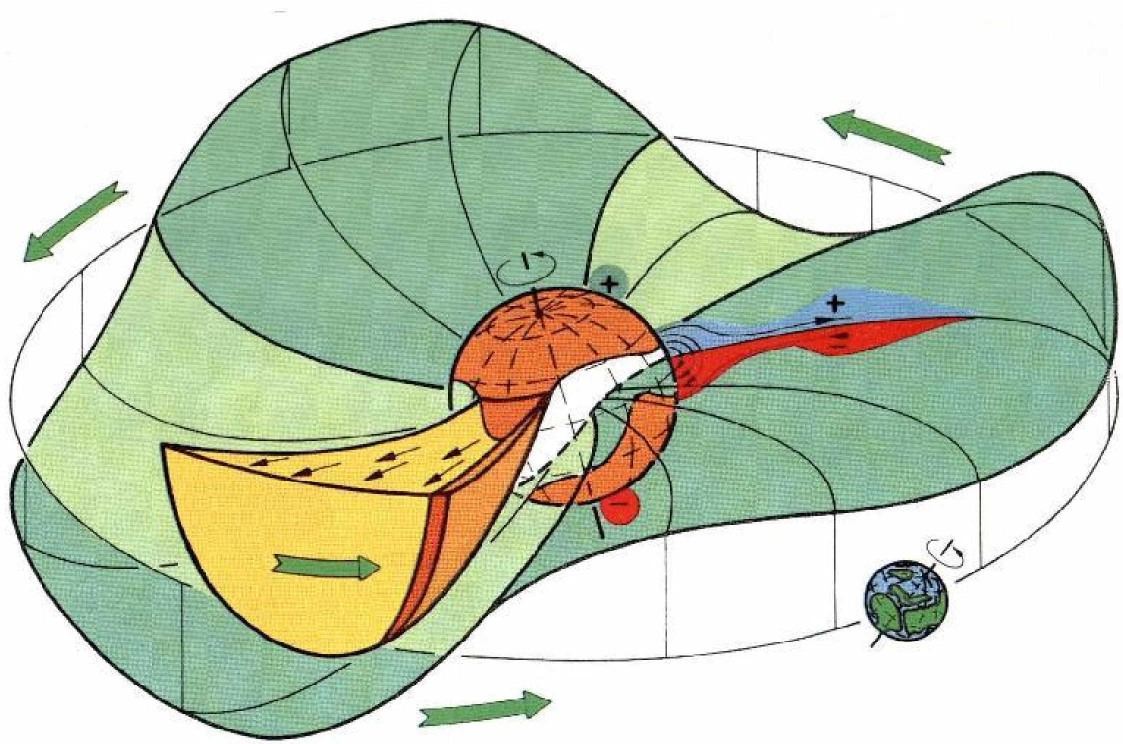
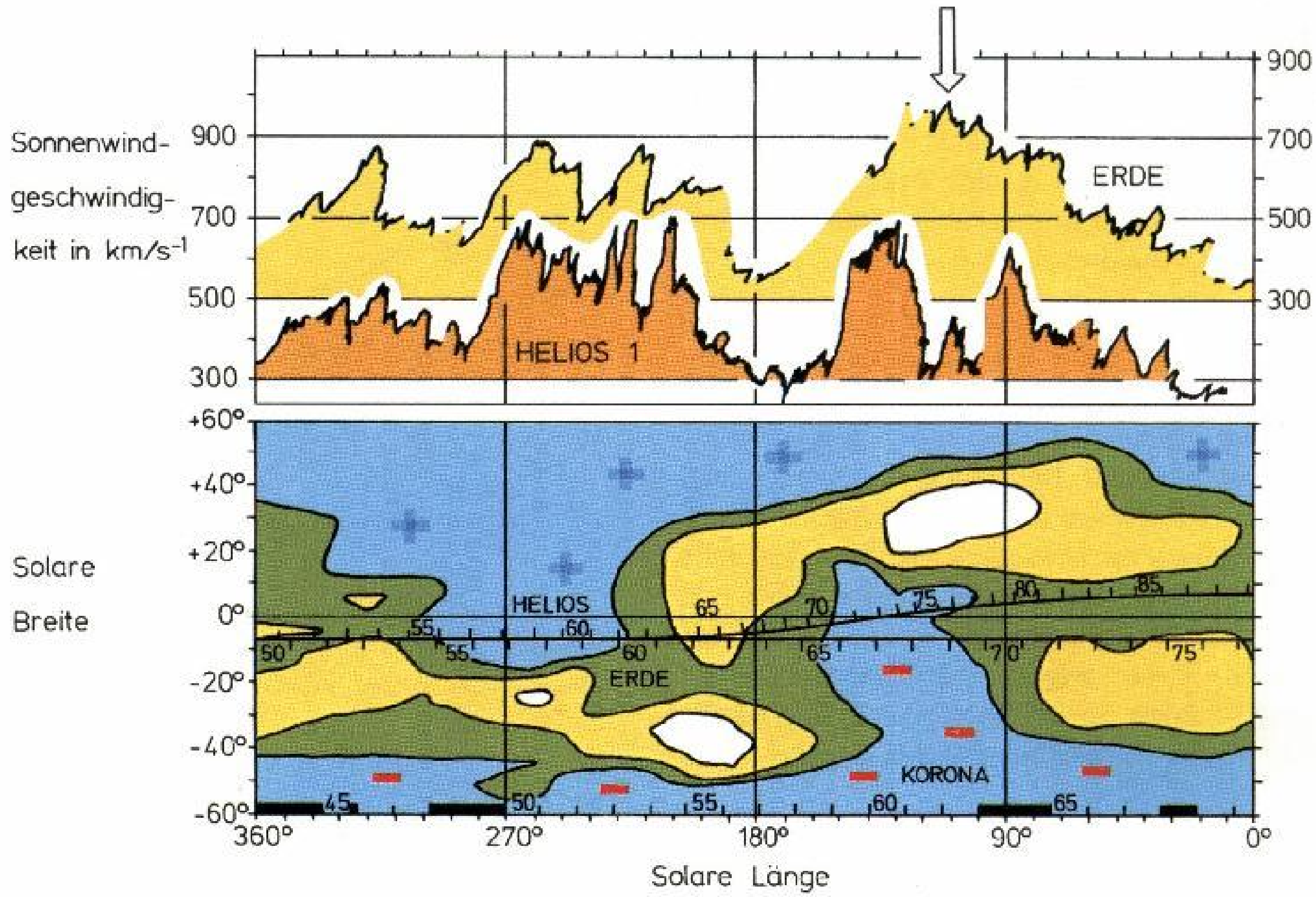


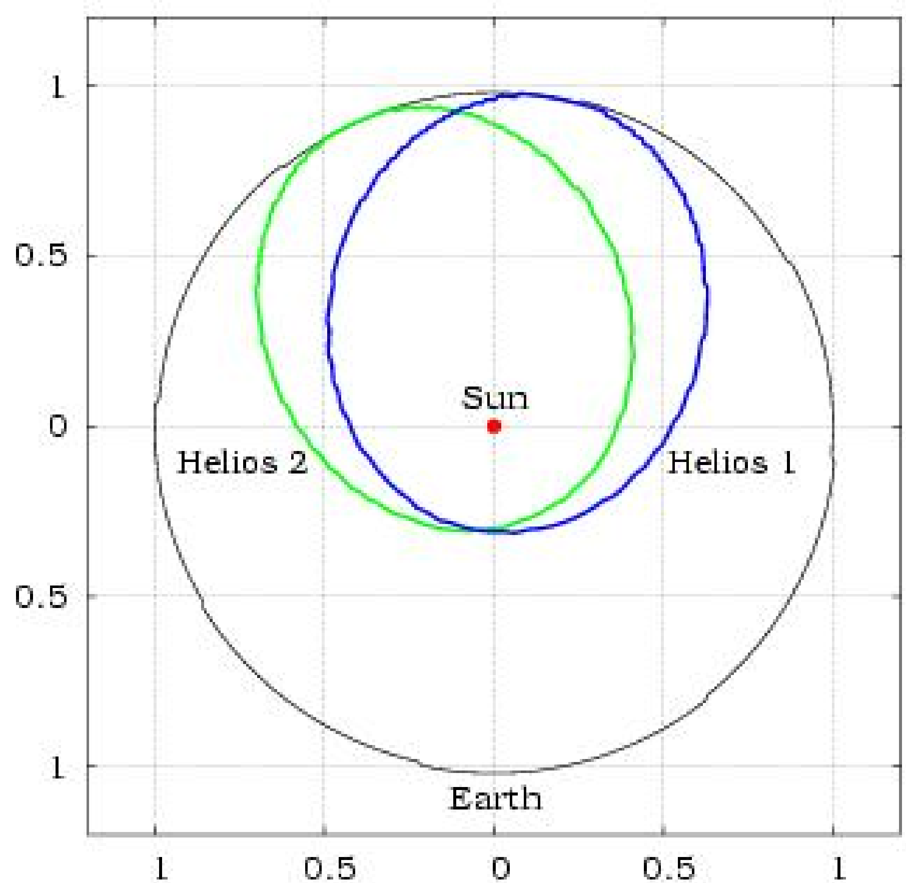
A schematic model of the solar wind ”ballerina”: the Sun’s two hemispheres are separated by a neutral layer of a form reminiscent of a ’ballerina’s skirt’. In the inner heliosphere the solar wind streams are of two forms called the slow ($\approx$ 400 km s^{-1}) and fast ($\approx$ 700 km s^{-1}). The fast wind is associated with coronal holes and is relatively uniform and stable, while the slow wind is quite variable, taken from (Schwenn and Rosenbauer, 1984).



Structures in the solar wind and their sources in the corona (solar map), taken from (Schwenn and Rosenbauer, 1984).

Helios Spacecraft

Orbits



Attractor Reconstruction

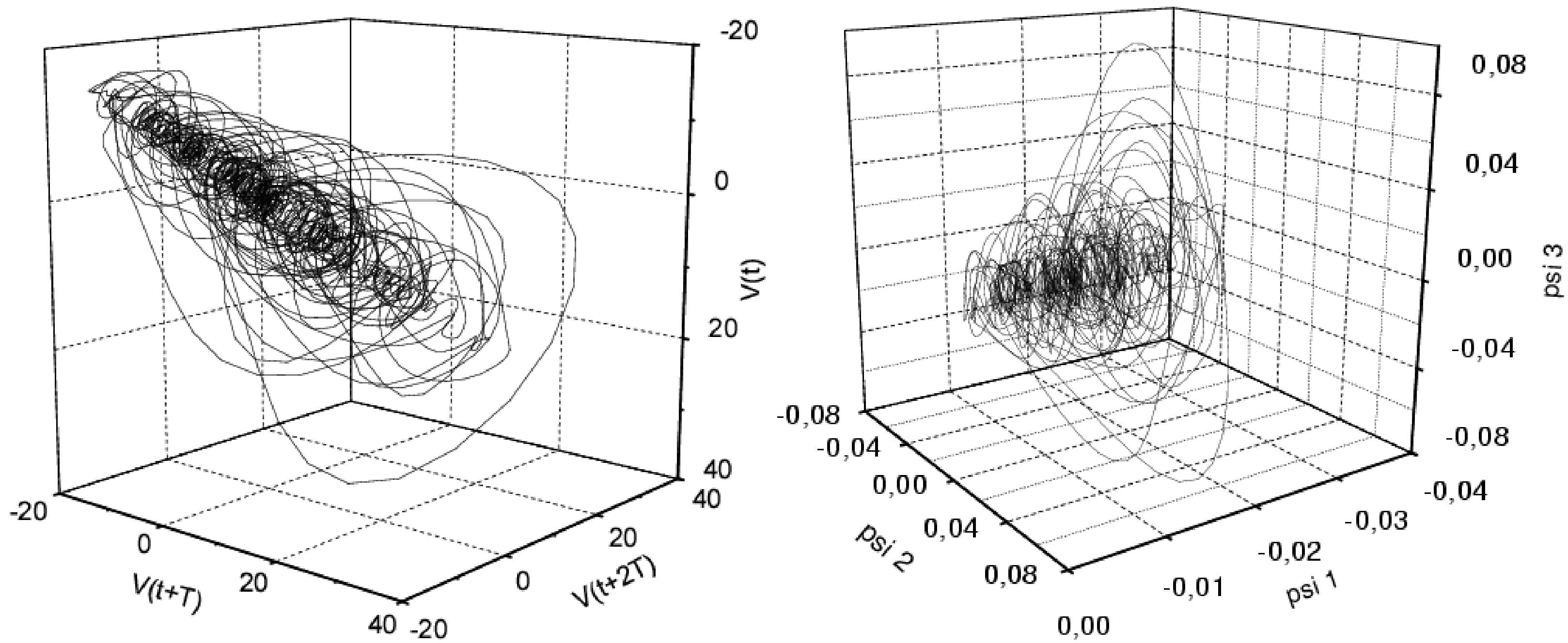


Fig. 2. The projection of the attractor onto the three-dimensional space, reconstructed from the detrended data, $T = 4\,\Delta t$, using (a) the moving average and also (b) the singular-value decomposition filters ($\Psi = U$), taken from (Macek, 1998).

Methods of Data Analysis

Generalized Dimensions

The generalized dimensions are important characteristics of *complex* dynamical systems. Since these dimensions are related to frequencies with which typical orbits in phase space visit different regions of the system, they can provide information about its dynamics.

The modern technique of nonlinear time series analysis allows to estimate the multifractal measure directly from a single time series.

Structure Functions Scaling

$S_u^q(l)$, *q*th order structure function ($q > 0$) in the inertial range ($\eta \ll l \ll L$)

$$S_u^q(l) = \langle |u(x+l) - u(l)|^q \rangle \sim l^{\xi(q)} \tag{1}$$

$u(x)$, a velocity component parallel to the longitudinal direction l
$\xi(q)$, a scaling exponent.

Energy Transfer Rate

$$\varepsilon_l \sim \frac{|u(x+l)-u(x)|^3}{l} \qquad \mu_i = \frac{\varepsilon_l}{\langle \varepsilon_L \rangle} \tag{2}$$

$$\sum_i \mu_i^q \sim l^{\tau(q)} \tag{3}$$

$$\tau(q) = (q-1)\; D_q \tag{4}$$

ε_l, a transfer of energy per unit time (and unit mass)
μ_i, probability measure of ith eddy in the d-dimensional physical space.

From Equations (1) to (4) we have (Tsang *et al.*, 2005):

$$\tau(q) = d(q-1) + \xi(3q) - q\xi(3) \tag{5}$$

Mutifractal Models for Turbulence

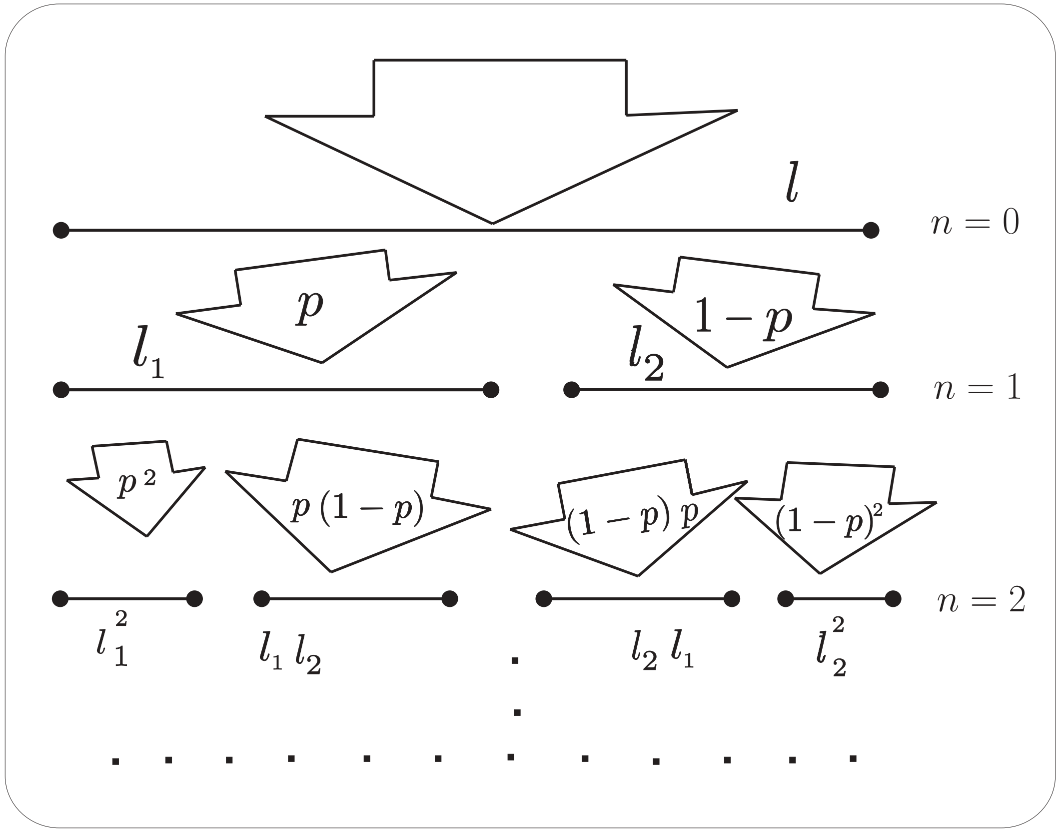


Fig. 1. Generalized two-scale Cantor set model for turbulence (Macek, 2007).

For the generalized self-similar weighted Cantor set (acting on the unit interval) we use the following partition function at n-th level of construction (Hentschel and Procaccia 1983; Halsey *et al.*, 1986)

$$\Gamma_n^q(l_1, l_2, p_1, p_2) = \left(\frac{p_1^q}{l_1^{\tau(q)}} + \frac{p_2^q}{l_2^{\tau(q)}} \right)^n = 1 \tag{6}$$

Parameters:

- $p_1 = p \leq 1/2$, natural invariant measure on the attractor of the system, the probability of visiting one region of the interval (the probability of visiting the remaining region is $p_2 = 1 - p$);
- $l_1 + l_2 \leq 1$, two nonuniform compression (dissipation) parameters (stretching and folding in the phase space).

Solutions

Transcendental equation (for $n \to \infty$)

$$\frac{p^q}{l_1^{\tau(q)}} + \frac{(1-p)^q}{l_2^{\tau(q)}} = 1 \tag{7}$$

Legendre transformation

$$\alpha(q) = \frac{\mathrm{d}\ \tau(q)}{\mathrm{d}q} \tag{8}$$

$$f(\alpha) = q\alpha(q) - \tau(q) \tag{9}$$

For $l_1 = l_2 = s$ and any q in Eq. (7) one has for the generalized dimension of the attractor (projected onto one axis)

$$(q-1)D_q = \frac{\ln[p^q + (1-p)^q]}{\ln s}. \tag{10}$$

No dissipation ($s = 1/2$):
the multifractal cascade $p-$model for fully developed turbulence,
the generalized weighted Cantor set (Meneveau and Sreenivasan, 1987).
The usual middle one-third Cantor set (without any multifractality):
$p = 1/2$ and $s = 1/3$.

The difference of the maximum and minimum dimension
(the least dense and most dense points on the attractor)

$$\Delta \equiv \alpha_{\max} - \alpha_{\min} = D_{-\infty} - D_{\infty} = \left| \frac{\log(1-p)}{\log l_2} - \frac{\log(p)}{\log l_1} \right| \tag{11}$$

In the limit $p \to 0$ this difference rises to infinity (degree of multifractality).

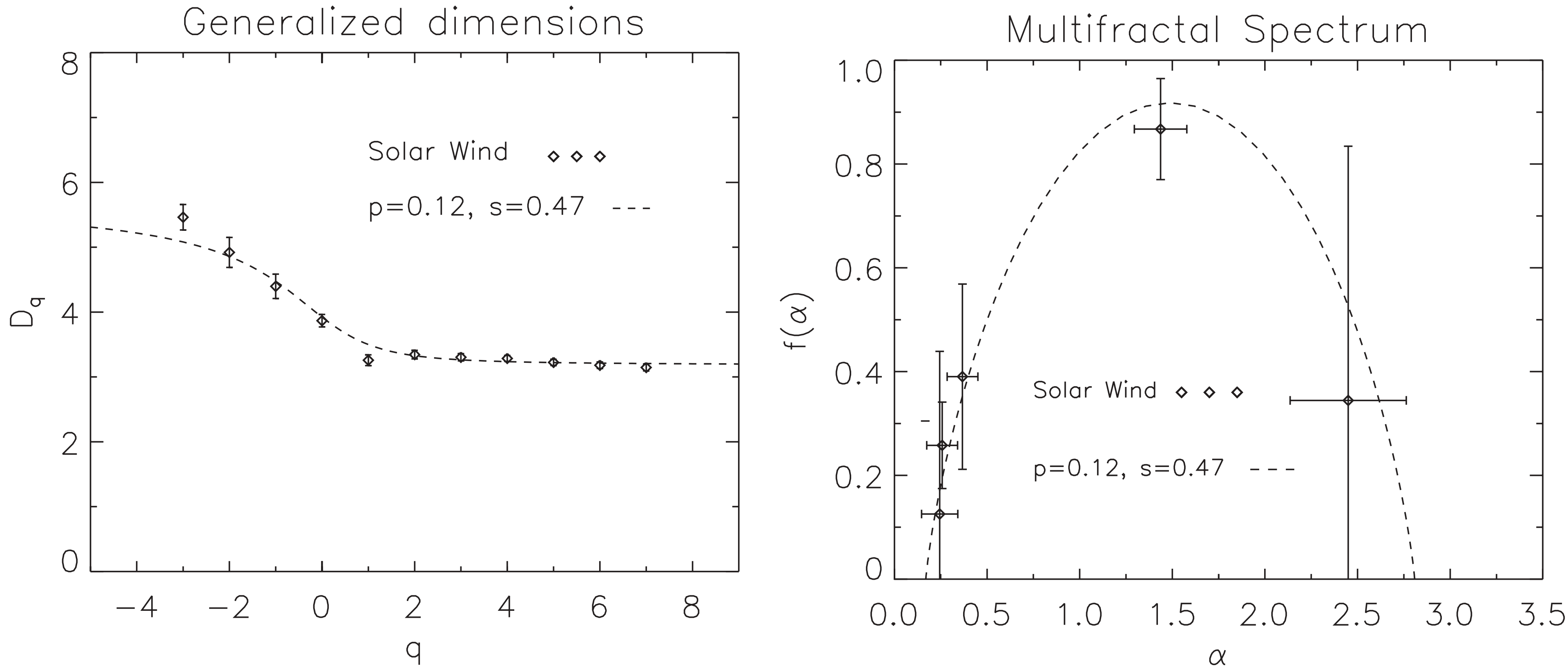


Fig. 3. (a) The generalized dimensions D_q in Equation (4) as a function of q. The correlation dimension is $D_2 = 3.4 \pm 0.1$. The values of $D_q + 3$ are calculated analytically for one-scale weighted Cantor set (baker's map) with $p = 0.12$ and $s = 0.47$ (dashed line). (b) The singularity spectrum $f(\alpha)$ as a function of α. The values of $f(\alpha)$ projected onto one axis for the weighted baker's map with the same parameters (dashed line), taken from (Macek, 2006).

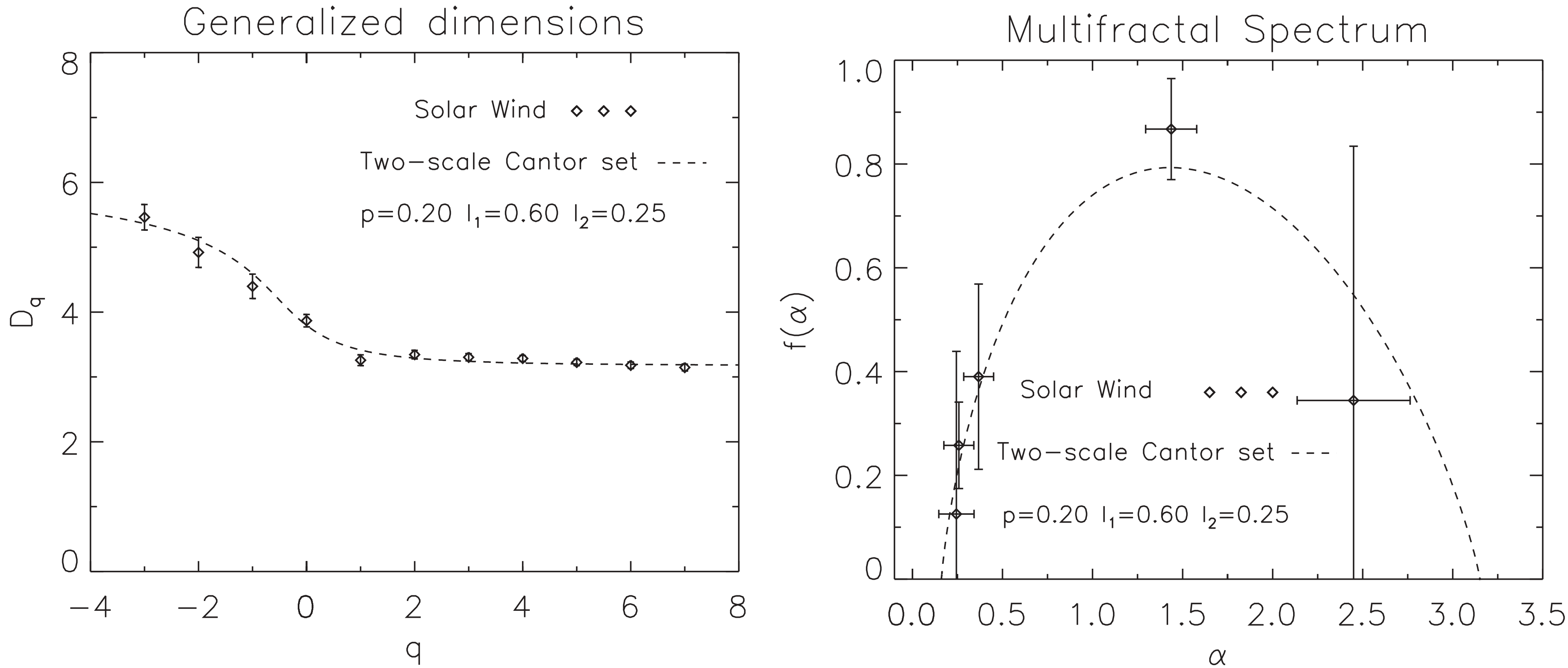


Fig. 4. (a) The generalized dimensions D_q in Equation (4) as a function of q. The values of $D_q + 3$ are calculated analytically for the weighted two-scale Cantor set with $p = 0.20$ and $l_1 = 0.60$, $l_2 = 0.25$ (dashed line). (b) The singularity spectrum $f(\alpha)$ as a function of α. The values of $f(\alpha)$ projected onto one axis for the weighted two-scale Cantor set with the same parameters (dashed line), taken from (Macek, 2007).

Data

Table 1: The time intervals (days) of Helios 2 data in 1976 for slow and fast solar wind streams measured at various distances from the Sun.

	~ 0.3 AU	~ 0.97 AU
Slow streams	99 - 102	105 - 108
Fast streams	26 - 29	21 - 24

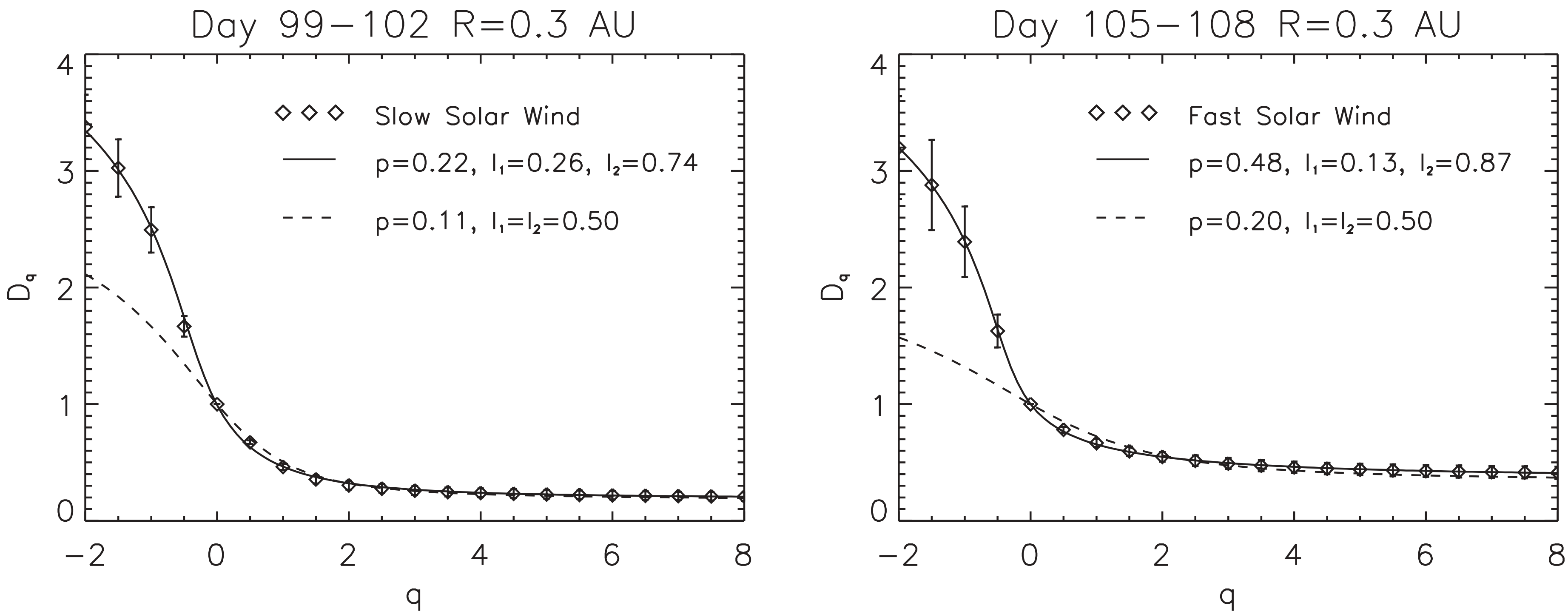


Fig. 5. The generalized dimensions D_q as a function of q. The values for one-dimensional turbulence are calculated for the generalized two-scale (continuous lines) model and the usual one-scale (dashed lines) p-model and fitted using the v_x radial velocity components (diamonds) for the slow (a) and fast (b) solar wind streams at distances of 0.3 AU (Macek and Szczepaniak, 2008).

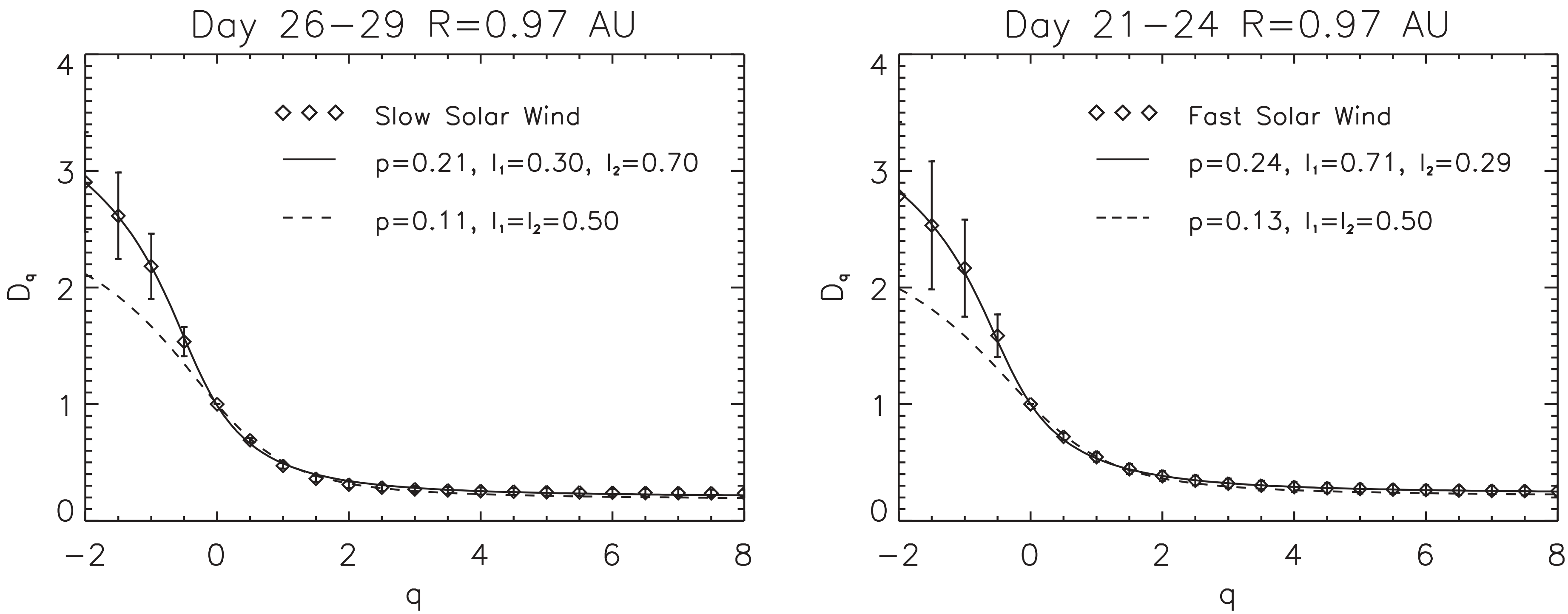


Fig. 5. The generalized dimensions D_q as a function of q. The values for one-dimensional turbulence are calculated for the generalized two-scale (continuous lines) model and the usual one-scale (dashed lines) p-model and fitted using the $u = v_x$ radial velocity components (diamonds) for the slow (c) and fast (d) solar wind streams at distances of 0.97 AU (Macek and Szczepaniak, 2008).

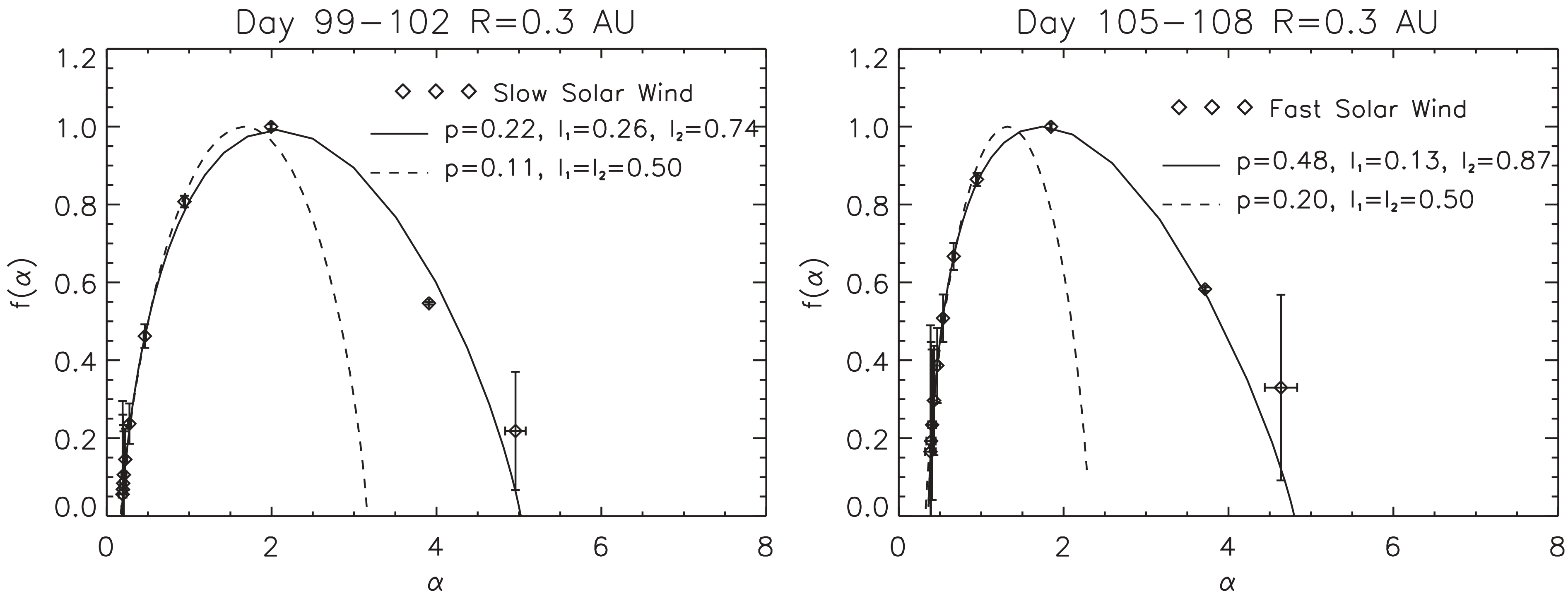


Fig. 6. The singularity spectrum $f(\alpha)$ as a function of α. The values for one-dimensional turbulence are calculated for the generalized two-scale (continuous lines) model and the usual one-scale (dashed lines) p-model and fitted using the $u = v_x$ radial velocity components (diamonds) for the slow (a) and fast (b) solar wind streams at distances of 0.3 AU (Macek, 2008).

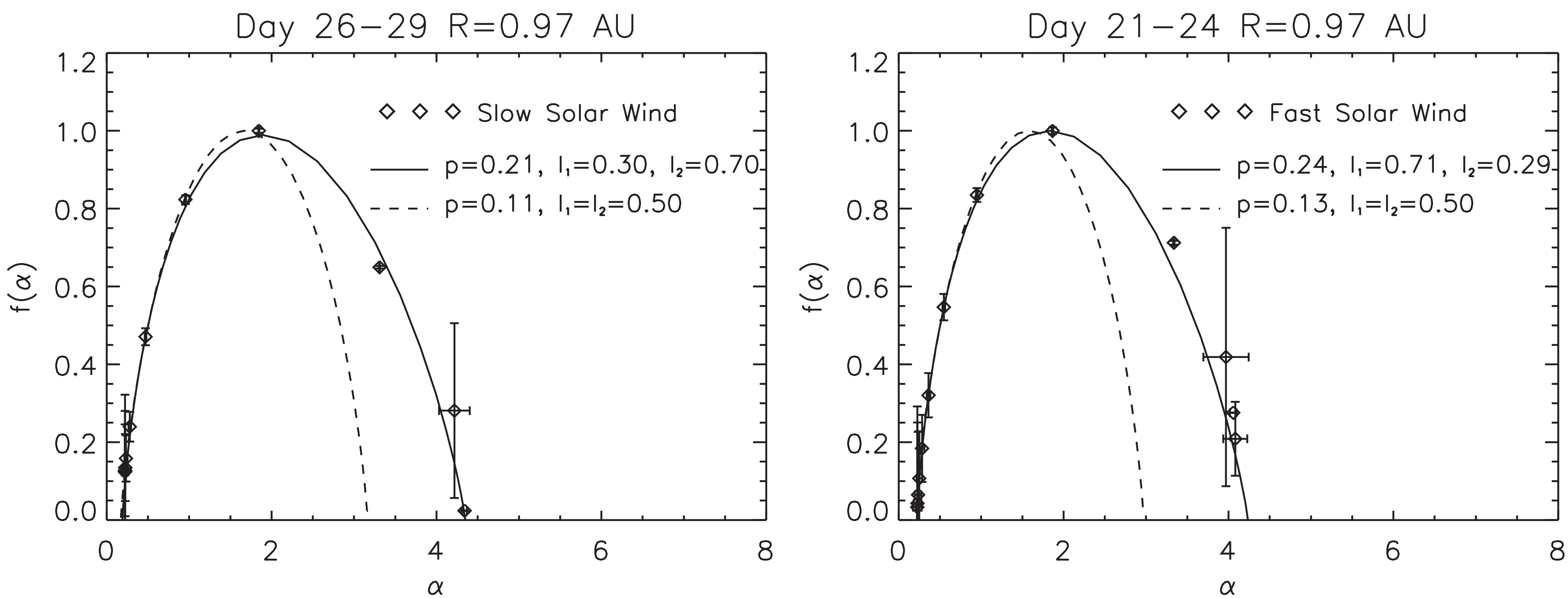


Fig. 6. The singularity spectrum $f(\alpha)$ as a function of α. The values for one-dimensional turbulence are calculated for the generalized two-scale (continuous lines) model and the usual one-scale (dashed lines) p-model and fitted using the $u = v_x$ radial velocity components (diamonds) for the slow (c) and fast (d) solar wind streams at distances of 0.97 AU (Macek, 2008).

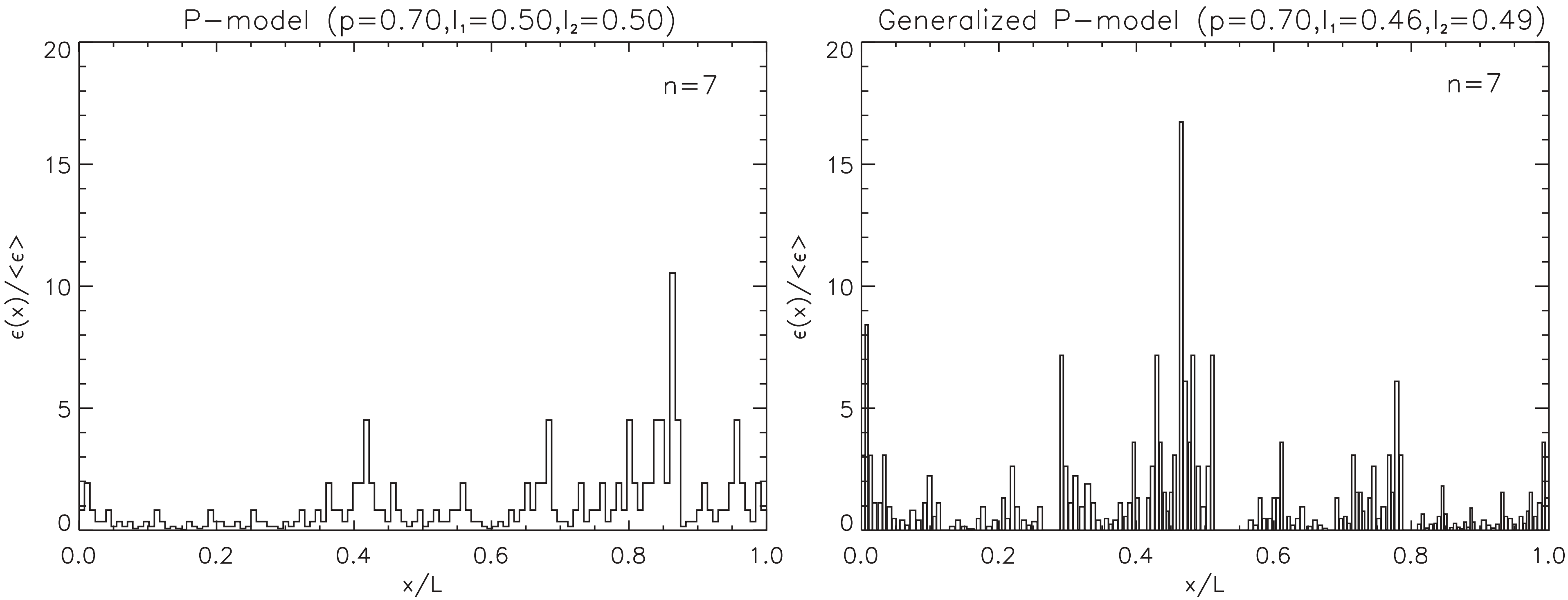


Fig. 7. The multifractal measure $\varepsilon/\langle\varepsilon\rangle$ on the unit interval for (a) the usual one-scale p-model (Meneveau and Sreenivasan, 1987) and (b) the generalized two-scale cascade model. Intermittent pulses are stronger for the model with two different scaling parameters (Macek and Szczepaniak, 2008).

The values of parameter p are related to the usual models, are based on the p-model of turbulence (e.g. Meneveau and Sreenivasan, 1987).

These values of p obtained here are roughly consistent with the fitted value in the literature both for laboratory and the solar wind turbulence, which is in the range $0.1 \leq p \leq 0.3$ (e.g., Burlaga, 1991; Carbone, 1993; Carbone and Bruno, 1996; Marsch *et al.*, 1996).

Conclusions

In this way, we have supported our conjecture that

- trajectories describing the system in the inertial manifold of phase space asymptotically approach the attractor of low-dimension (Macek, 1998).
- The obtained multifractal spectrum of this attractor is consistent with that for the multifractal measure on the generalized weighted two-scale Cantor set, which is a strange attractor that exhibits stretching and folding properties leading to sensitive dependence on initial conditions (Macek, 2006, 2007).
- The values of the parameters fitted for $l_1 + l_2 = 1$ and $p \sim 10^{-1}$, demonstrates small dissipation of the complex solar wind dynamical system and shows that some parts of the attractor in phase space are visited at least one order of magnitudes more frequently than other parts.

- We have also studied the inhomogeneous rate of the transfer of the energy flux indicating multifractal and intermittent behaviour of solar wind turbulence in the inner heliosphere.
- Basically, the generalized dimensions for solar wind are consistent with the generalized *p*-model for both positive and negative q, but rather with different scaling parameters for sizes of eddies, while the usual p-model can only reproduce the spectrum for $q \geq 0$. We have demonstrated that a much better agreement of the two-scale model with the real data is obtained, especially for $q < 0$.
- We also show that intermittent pulses are stronger for the model with asymmetric scaling.
- In general, the proposed generalized two-scale weighted Cantor set model should also be valid for non space filling turbulence. Therefore we propose this new cascade model describing intermittent energy transfer for analysis of turbulence in various environments.

Epilogue

Thus these results provide supporting evidence for **multifractal** structure of the solar wind in the inner heliosphere.

This means that the observed **intermittent** behavior of the solar wind's velocity and Alfvénic fluctuations results from intrinsic *nonlinear* dynamics rather than from random external forces.

The multifractal structures, convected by the solar wind, might probably be related to the complex topology shown by the magnetic field at the source regions of the solar wind.

Thank you!

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