Chaos, Strange Attractors, and Intermittency in the Generalized Lorenz Model

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Abstract

We consider a low-dimensional model of convection in a horizontally magnetized layer of a viscous fluid heated from below. We analyze in detail the stability of hydromagnetic convection influenced by the induced magnetic field for a wide range of two control parameters. Namely, when changing the initially applied temperature difference or magnetic field strength, one can see transitions from regular to irregular long-term behavior of the system, switching between chaotic, periodic, and equilibrium asymptotic solutions. It is worth noting that owing to the induced magnetic field a transition to hyperchaotic dynamics is possible for some parameter values of the model. In particular, we discuss in detail irregular behavior of the system including new strange attractors, also in a hyperchaotic regime and new types of bifurcations leading to intermittency. We also reveal new features of the generalized Lorenz model, including both type I and III intermittency.

Plan of Presentation

1. Introduction
   - Rayleigh-Bénard Convection and Lorenz Model
   - Hydromagnetic Convection
2. The Model of Hydromagnetic Convection
   - Basic Equations
   - Approximations
   - Generalized Lorenz Model for a Magnetized Fluid
3. Analysis of the Model
   - Dynamical Behavior in Control Parameters Space
   - New Strange Attractors
   - Intermittency
   - Hyperchaotic Convection
4. Possible Applications
5. Conclusions

Hydromagnetic Convection

Dynamics of irregular flows in viscous fluids is still not sufficiently well understood. Newly published paper in Physical Review Letters sheds light on hydromagnetic convection.

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It appears that behavior of this system can rather be complex: from equilibrium or regular (periodic) motion, through intermittency (where irregular and regular motions are intertwined) to nonperiodic behavior. Two types of such nonperiodic flows are possible, namely chaotic and hyperchaotic motions. As discovered by Lorenz (1963), deterministic chaos exhibits sensitivity to initial conditions leading to unpredictability of the long-term behavior of the system (butterfly effect).

Obviously, hyperchaos is a more complex nonperiodic flow, which is now discovered in the generalized Lorenz model previously proposed by the authors in 2010. The results of the present paper illustrate how all these complex motions can be studied by analyzing this simple model.

In particular, it is shown that various kinds of complex behavior are closely neighbored depending on two control parameters of the model.

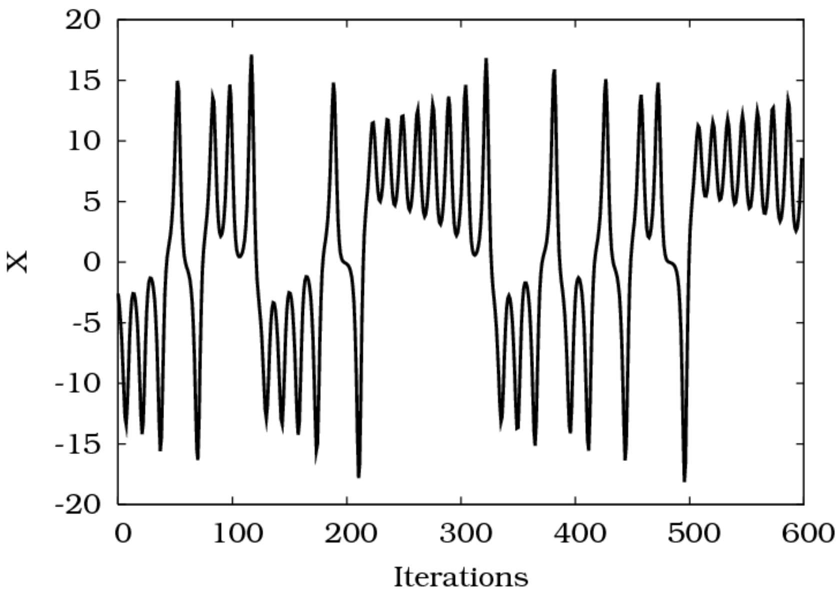
Naturally, the convection appears in plasmas, where electrically charged particles interact with the magnetic field. Therefore, the obtained results could be important for explaining dynamical processes in solar sunspots, planetary and stellar liquid interiors, and possibly for plasmas in nuclear fusion devices. It worth noting that in order to get similar information from direct numerical simulations one would require many years of instantaneous computations using tremendous computational resources.

Lorenz Model

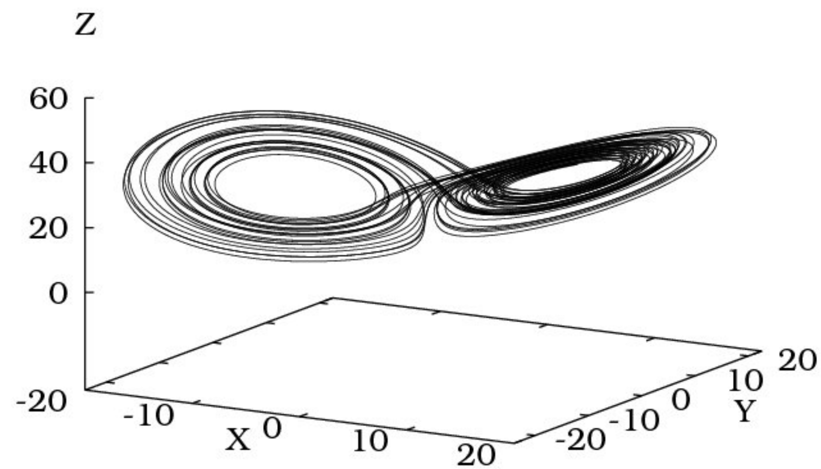
$$\dot{X} = \sigma(Y - X)$$
$$\dot{Y} = -XZ + rX - Y$$
$$\dot{Z} = XY - bZ$$

Parameters:
$r = 28$, $\sigma = 10$, $b = 8/3$

Time series for X

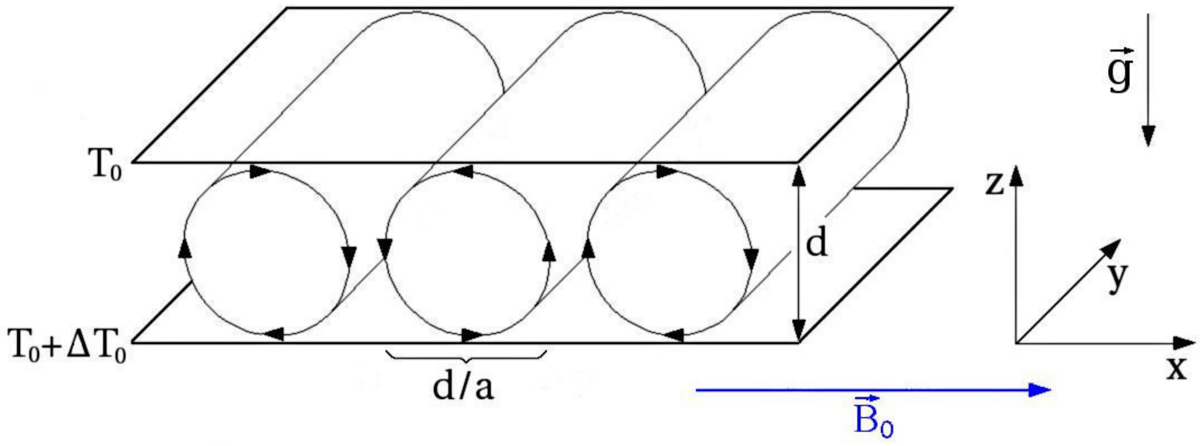


Attractor



- Macek W. M., Strumik, M., Model for hydromagnetic convection in a magnetized fluid, Physical Review E **82**, 027301, 2010, doi = 10.1103/PhysRevE.82.027301.
- Macek, W. M., Strumik, M., Hyperchaotic intermittent convection in a magnetized viscous fluid, Physical Review Letters **112**, 074502, 2014. doi: 10.1103/PhysRevLett.112.074502, http://link.aps.org/abstract/PRL/v112/e074502.

Convection in a Magnetized Fluid



$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla\left(p + \frac{\mathbf{B}^2}{2\mu_o}\right) + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu_o\rho} + \nu \triangle \mathbf{v} + \mathbf{f}, \tag{1}$$

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\cdot\nabla)\mathbf{v} + \eta \triangle \mathbf{B}, \tag{2}$$

$$\frac{dT}{dt} = \kappa \triangle T. \tag{3}$$

Convection in a Magnetized Fluid

Additional conditions

$$\nabla \cdot \mathbf{v} = 0,$$
$$\nabla \cdot \mathbf{B} = 0$$

allow to define

$$\mathbf{v} = \nabla \times \Psi,$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

a stream (potential) function ψ for the flow $\mathbf{v}$:

$$\Psi = \{0, \psi(x,z,t), 0\}$$

and a (vector) potential $\mathbf{A}$ for the embedded magnetic field $\mathbf{B}$:

$$\mathbf{A}/(\mu_o \rho_o)^{1/2} \; = \; \{0, \alpha(x,z,t) - \upsilon_{\mathrm{A}o} z, 0\}$$

where $\upsilon_{\mathrm{A}o} = B_o/(\mu_o \rho_o)^{1/2}$ is the initial Alfvén speed.

Approximation used: $(\mathbf{B} \cdot \nabla)\mathbf{v} \approx (\mathbf{B_o} \cdot \nabla)\mathbf{v}$.

Rayleigh-Bénard Convection in a Magnetized Fluid

Double asymmetric Fourier representation:

$$\psi(x,z,t) = \sqrt{2}\,\frac{1+a^2}{a}\,\kappa\,X(t)\ \sin\left(\frac{\pi a}{h}x\right)\ \sin\left(\frac{\pi}{h}z\right),$$

$$\theta(x,z,t) = \frac{R_c\delta T_0}{\pi R_a}\left[\sqrt{2}\,Y(t)\cos\left(\frac{\pi a}{d}x\right)\sin\left(\frac{\pi}{d}z\right) - Z(t)\sin\left(\frac{2\pi}{d}z\right)\right],$$

$$\alpha(x,z,t) = \sqrt{2}\,\frac{1+a^2}{a}\,\kappa\,W(t)\ \cos\left(\frac{\pi a}{h}x\right)\ \sin\left(\frac{\pi}{h}z\right).$$

The velocity field is described by $\psi(x,z,t)$ [variable $X(t)$],
the temperature gradient by $\theta(x,z,t)$ [variables $Y(t), Z(t)$],
and the induced magnetic field by $\alpha(x,z,t)$ [variable $W(t)$].

Lorenz Model for a Magnetized Fluid

Using those approximations fluid dynamics can be described by a simple set of four ordinary differential equations

$$\dot{X} = -\sigma X + \sigma Y - \omega_0 W, \quad (4)$$

$$\dot{Y} = -XZ + rX - Y, \quad (5)$$

$$\dot{Z} = XY - bZ, \quad (6)$$

$$\dot{W} = \omega_0 X - \sigma_{\mathrm{m}} W, \quad (7)$$

where a dot denotes an orinary derivative with respect to the normalized time $t' = (1+a^2)\ \kappa(\pi/h)^2\ t$, using a geometrical factor $b = 4/(1+a^2)$.

Control parameter $r = R_{\mathrm{a}}/R_{\mathrm{c}}$;
Rayleigh number $R_{\mathrm{a}} = g\beta h^3 \delta T/(\nu\kappa)$, critical number $R_{\mathrm{c}} = (1+a^2)^3(\pi^2/a)^2$.
Magnetic control parameter
$\omega_o = \upsilon_{\mathrm{A}o}/\upsilon_o$;
$\upsilon_{\mathrm{A}o} = B_o/(\mu_o\rho_o)^{1/2}$, $\upsilon_o = 4\pi\kappa/(abh)$
Prandtl number $\sigma = \nu/\kappa$;
Magnetic Prandtl number ν/η, $\sigma_{\mathrm{m}} = \eta/\kappa$.

Lorenz Model for a Magnetized Fluid

Combining the set of the generalized Lorenz system we can write

$$\ddot{X} + \sigma \dot{X} + (\sigma r - \omega_o^2) X = -\sigma (Y + XZ) + \sigma_{\mathrm{m}} \omega_o W,$$

$$\ddot{W} + \sigma_{\mathrm{m}} \dot{W} + \omega_o^2 W = \sigma \omega_o (Y - X).$$

Hence formally both variables X and W satisfy the equations of two familiar damped linear oscillators with nonlinear driving forces. Moreover, we can see that the coupling between X, W and Y, Z is enhanced owing to the magnetic field $\mathbf{B}$. Obviously, when $\omega_o = 0$ this coupling ceases and the variable W is damped by the magnetic viscosity.

Fixed Points (Equilibrium)

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{F}(\mathbf{x}^*) = \mathbf{0}, \quad \mathbf{x}^* = \{X^*, Y^*, Z^*, W^*\}$$

$C^0 = \{0, 0, 0, 0\}$,
$C^\pm = \{\pm d/\sqrt{1+e}, \pm\, d\sqrt{(1+e)}, r - (1+e), \pm(\sigma/\omega_o) de/\sqrt{1+e}\}$,
where $d = \sqrt{b((r-1)-e)}$, and $e = \omega_o^2/(\sigma\, \sigma_{\mathrm{m}})$.

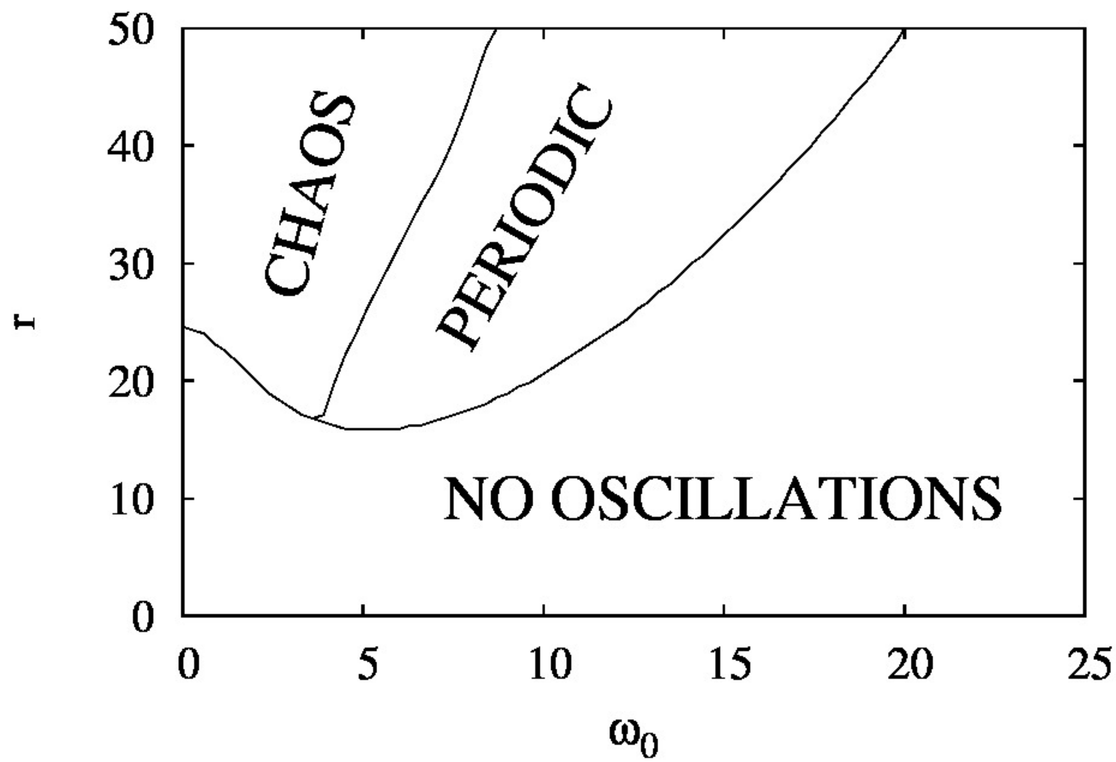
C^0 stable for $\quad 0 \le r < r_o$,
$C^\pm$ stable for $\quad r_o \le r < r_H$,
$r_o = 1 + e$ is a critical value for the onset of convection,
$r = r_H$ is a critical value, where a Hopf bifurcation takes place.

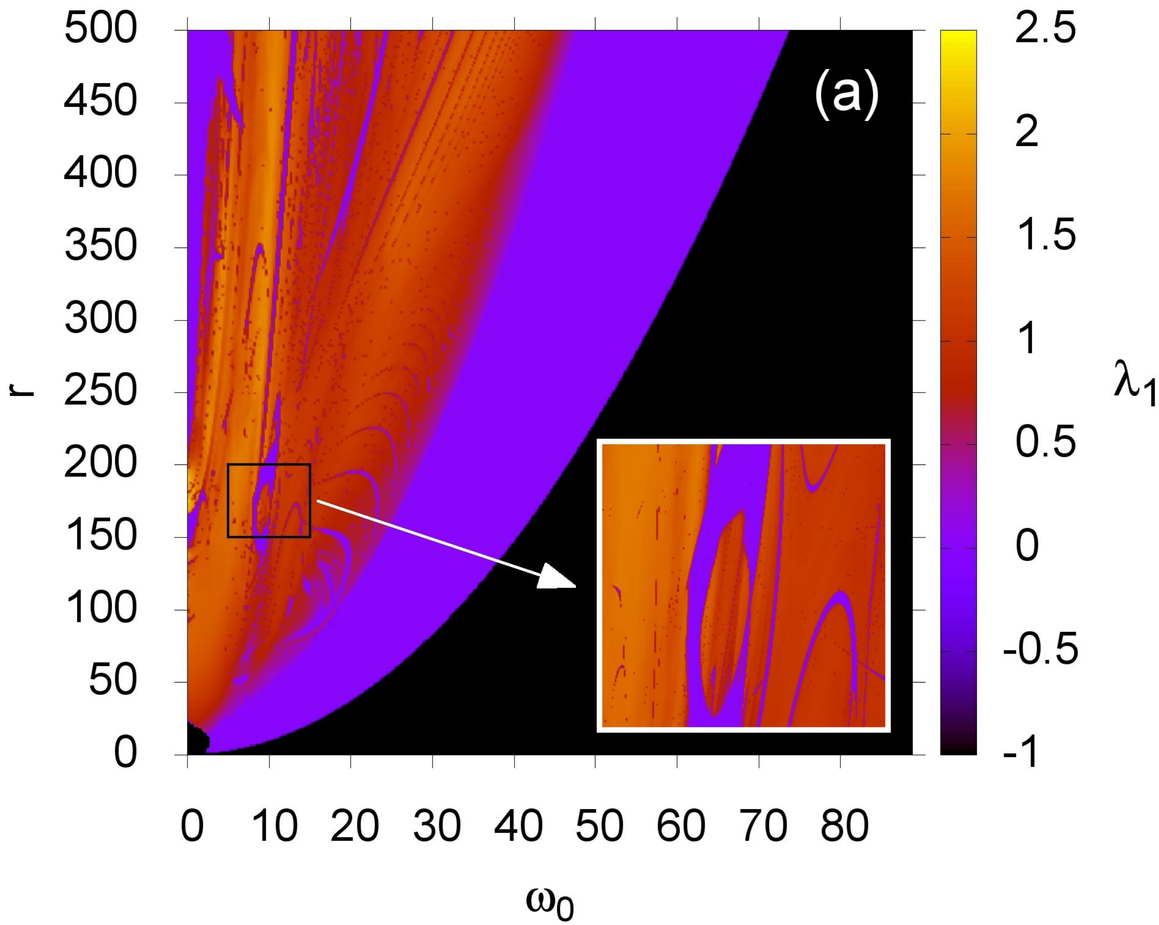
The critical number r_o for the onset of convection increases with the magnetic field, thus the magnetic field should stabilize the convection as regards to the appearance of convective rolls.

However, if we consider oscillations of the convection rolls, the influence of the magnetic field is more intricate.

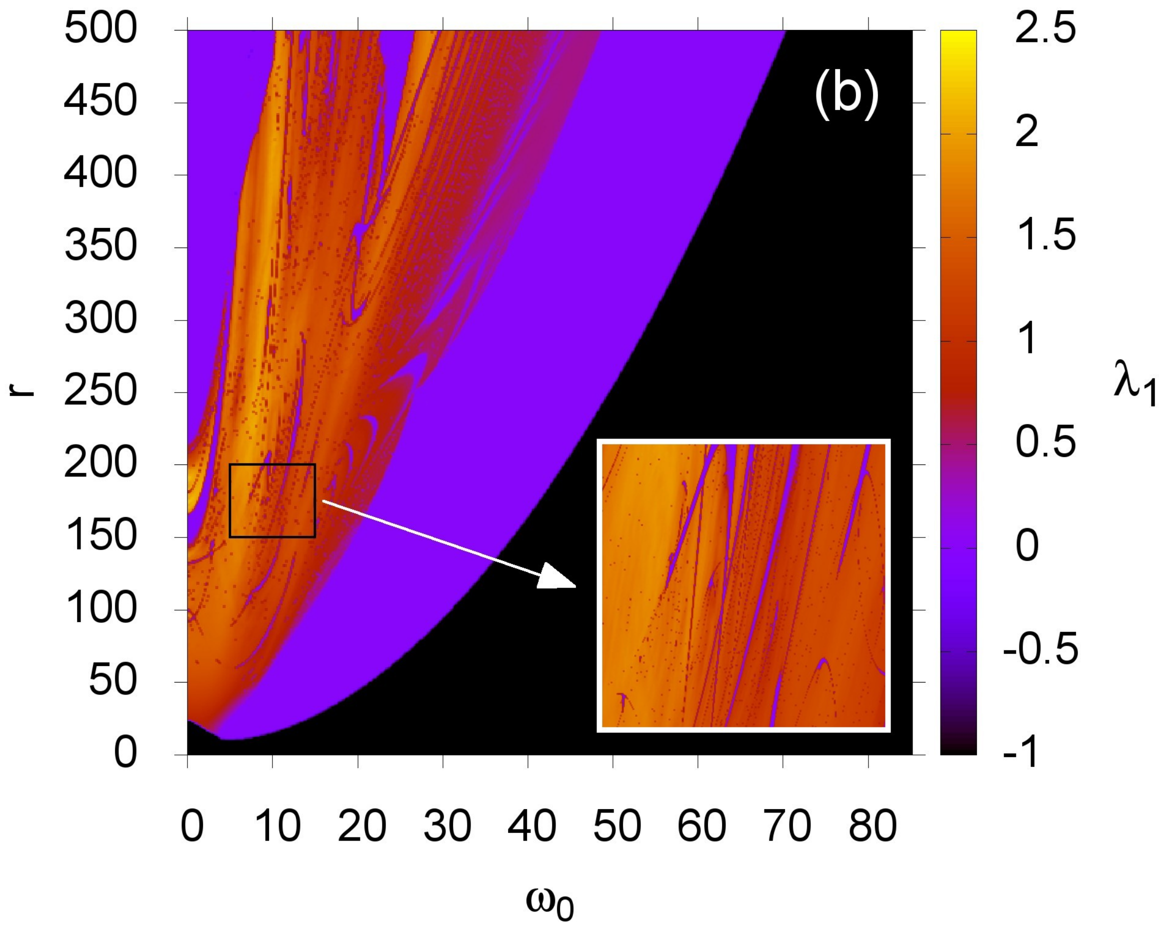
Long-term Behavior Depending on Control Parameters

$$r = R_{\mathrm{a}}/R_{\mathrm{c}}, \quad R_{\mathrm{a}} = g\beta h^3 \delta T/(\nu\kappa),\ R_{\mathrm{c}} = (1+a^2)^3(\pi^2/a)^2$$
$$\omega_o = \upsilon_{\mathrm{A}o}/\upsilon_o, \quad \upsilon_{\mathrm{A}o} = B_o/(\mu_o\rho_o)^{1/2},\ \upsilon_o = 4\pi\kappa/(abh)$$

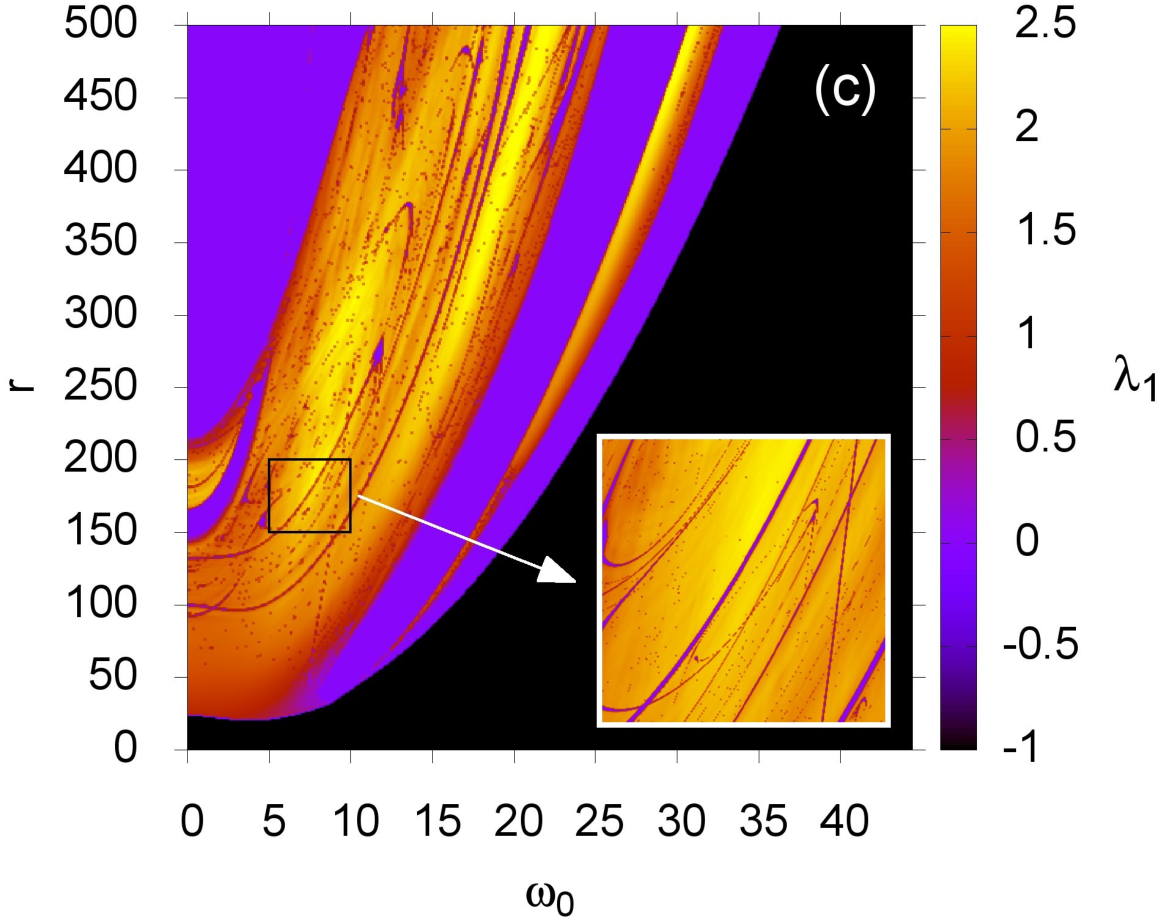




Dependence of the largest Lyapunov exponent λ_1 (color-coded) on ω_0 and r parameters of the generalized Lorenz model for (a) $\sigma_{\mathrm{m}} = 0.1$. Other parameters of the system have fixed values: $\sigma = 10$, $b = 8/3$. Convergence of the solutions of Eqs. (4)–(7) to fixed points ($\lambda_1 < 0$) is shown in black, to periodic solutions ($\lambda_1 = 0$) – in violet/blue color (see the color bar for $\lambda_1 = 0$), to chaotic solutions ($\lambda_1 > 0$) – in a color, consistently with the color bar scale, from violet/blue to yellow. For the panel an enlargement of the region bounded by black lines is shown in the right-bottom part of plots. Fine structures are shown in the inset (Macek and Strumik, 2014).



Dependence of the largest Lyapunov exponent λ_1 (color-coded) on ω_0 and r parameters of the generalized Lorenz model for (b) $\sigma_\mathrm{m} = 1$. Other parameters of the system have fixed values: $\sigma = 10$, $b = 8/3$. Convergence of the solutions of Eqs. (4)–(7) to fixed points ($\lambda_1 < 0$) is shown in black, to periodic solutions ($\lambda_1 = 0$) – in violet/blue color (see the color bar for $\lambda_1 = 0$), to chaotic solutions ($\lambda_1 > 0$) – in a color, consistently with the color bar scale, from violet/blue to yellow. For the panel an enlargement of the region bounded by black lines is shown in the right-bottom part of plots. Fine structures are shown in the inset (Macek and Strumik, 2014).



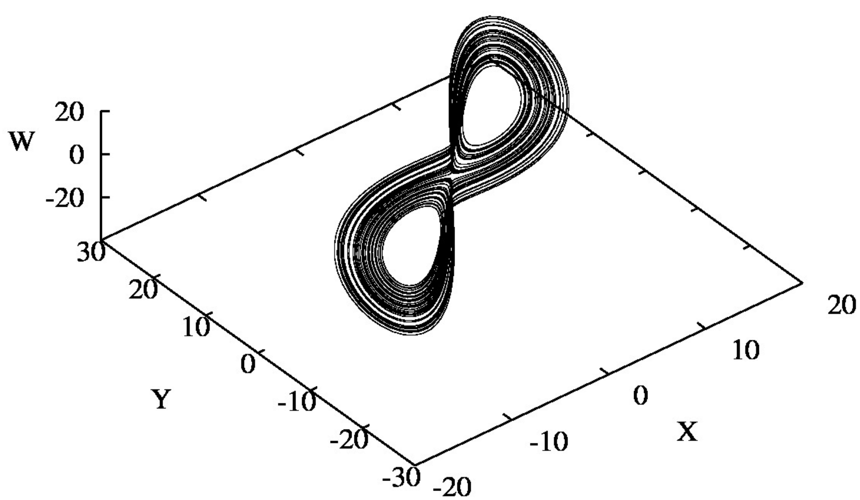
Dependence of the largest Lyapunov exponent λ_1 (color-coded) on ω_0 and r parameters of the generalized Lorenz model for (b) $\sigma_{\rm m} = 3$. Other parameters of the system have fixed values: $\sigma = 10$, $b = 8/3$. Convergence of the solutions of Eqs. (4)–(7) to fixed points ($\lambda_1 < 0$) is shown in black, to periodic solutions ($\lambda_1 = 0$) – in violet/blue color (see the color bar for $\lambda_1 = 0$), to chaotic solutions ($\lambda_1 > 0$) – in a color, consistently with the color bar scale, from violet/blue to yellow. For the panel an enlargement of the region bounded by black lines is shown in the right-bottom part of plots. Fine structures are shown in the inset (Macek and Strumik, 2014).

Strange Attractors for a Magnetized Fluid

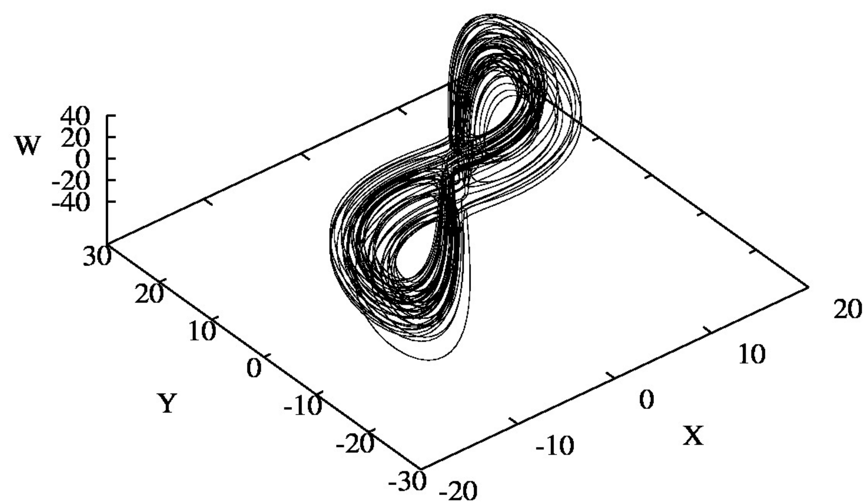
$$
\begin{aligned}
\dot{X} &= -\sigma X + \sigma Y - \omega_o W, \\
\dot{Y} &= -XZ + rX - Y, \\
\dot{Z} &= XY - bZ, \\
\dot{W} &= \omega_o X - \sigma_{\mathrm{m}} W,
\end{aligned}
$$

Standard parameters: $r = 28$, $\sigma = 10$, and $b = 8/3$

$\omega_o = 1$, $\sigma_{\mathrm{m}} = 20$

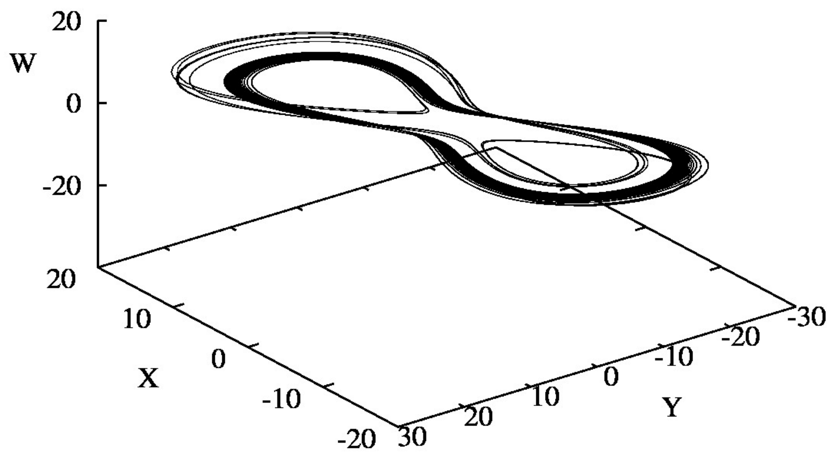


$\omega_o = 1$, $\sigma_{\mathrm{m}} \approx 0$

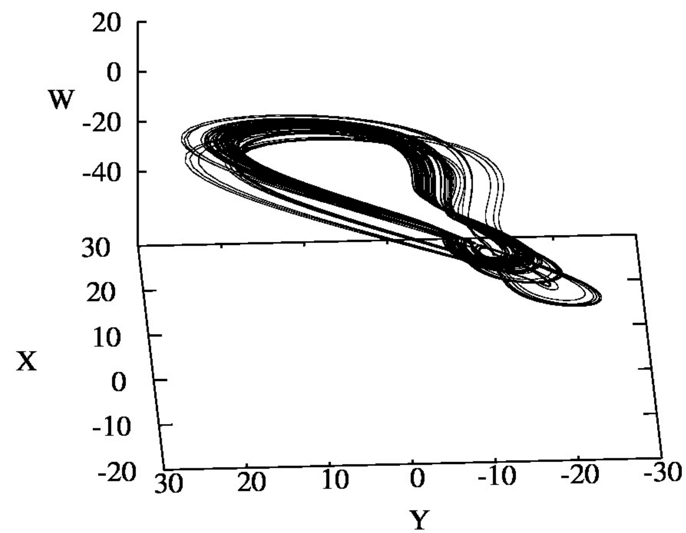


Strange Attractors for a Magnetized Fluid

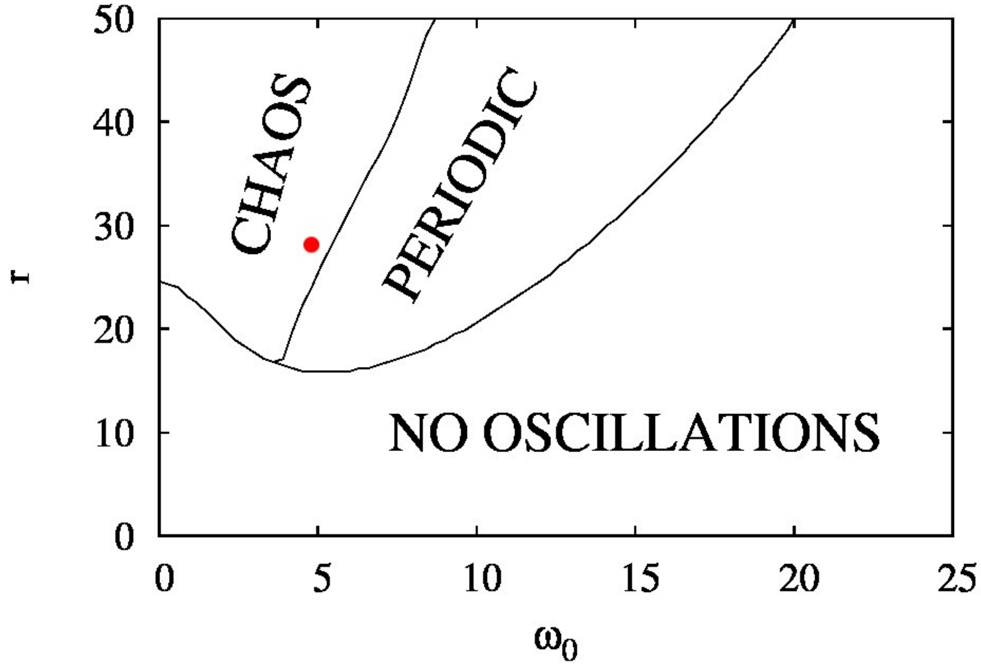
$\omega_o = 6,\ \sigma_{\mathrm{m}} = 2$

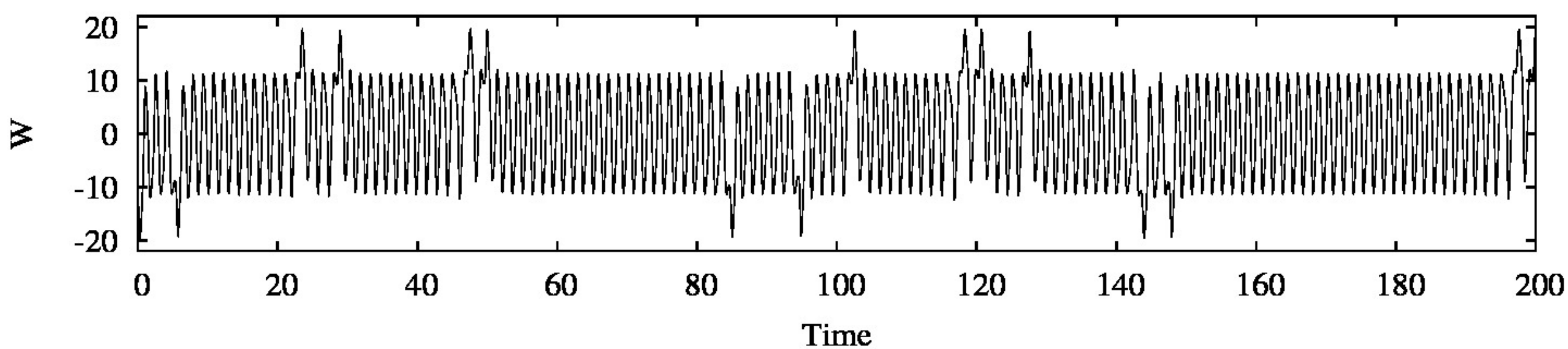


$\omega_o = 5,\ \sigma_{\mathrm{m}} \approx 0$



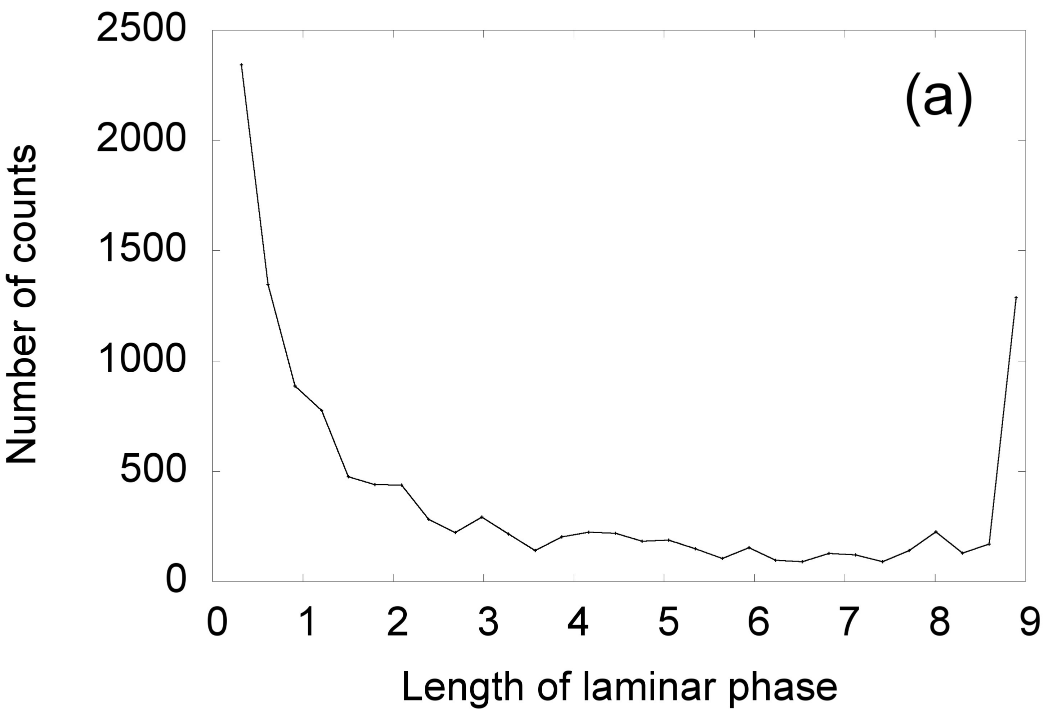
Intermittent Behavior





Intermittent Behavior

Type I intermittency (identified using Poincaré map based on Y values taken for X=0 plane crossings). Distribution of lengths τ of laminar phases for a deviation $\varepsilon = p - p_T > 0$ from a critical value for bifurcation p_T



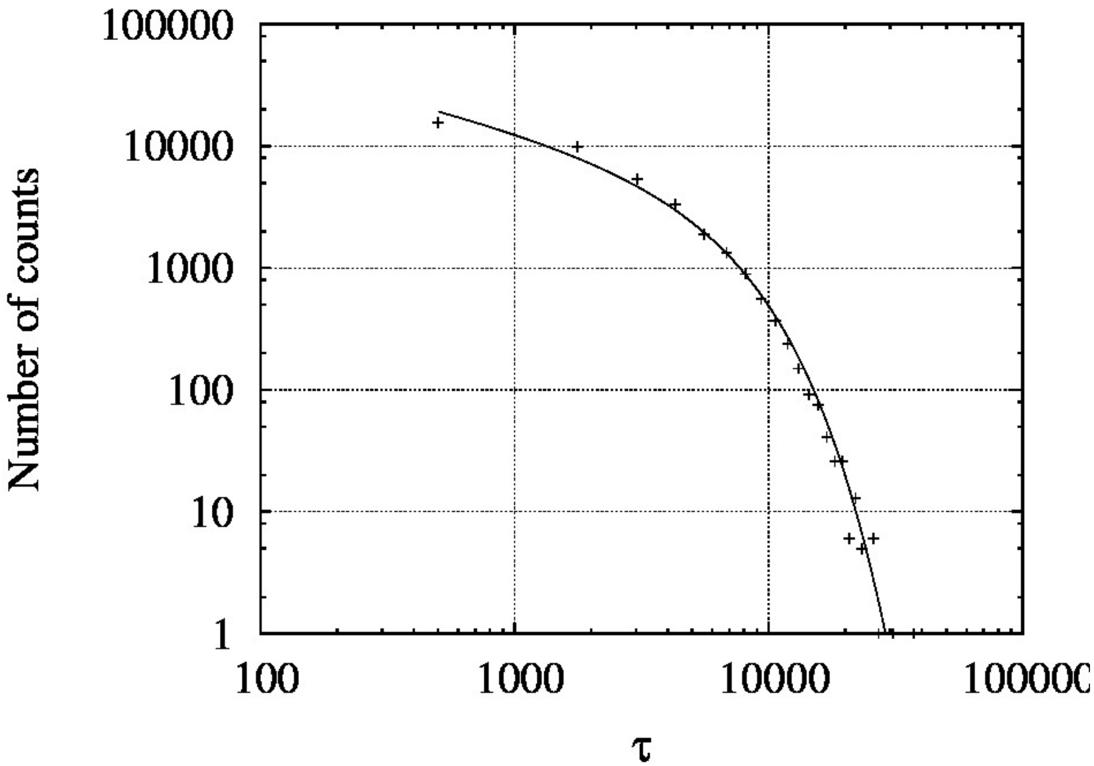
$$P(\tau) = \frac{\varepsilon}{2c}\left\{1 + \tan^2\left\{\arctan\left[\frac{c}{(\varepsilon/u)^{1/2}}\right] - \tau(\varepsilon u)^{1/2}\right\}\right\} \quad (8)$$

$$\text{average} < \tau > \sim \varepsilon^{-1/2}$$

U-shape and finite value of maximum length of laminar phase

Intermittent Behavior

Type III intermittency (identified using Poincaré map based on Y values taken for X=0 plane crossings). Distribution of lengths τ of laminar phases for a deviation $\varepsilon = p - p_T > 0$ from a critical value for bifurcation p_T



$$P(\tau) \propto \frac{\varepsilon^{3/2} e^{4\varepsilon\tau}}{(e^{4\varepsilon\tau} - 1)^{3/2}}$$

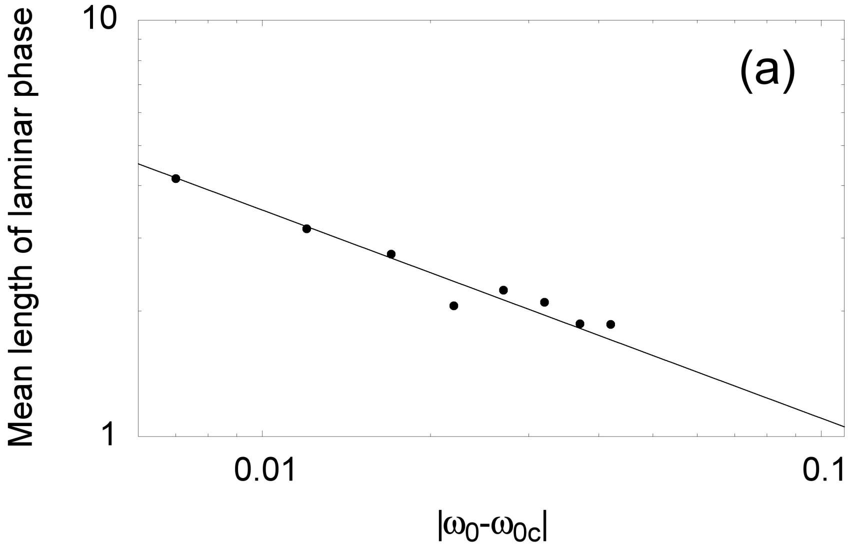
$$\text{average} < \tau > \sim \varepsilon^{-1}$$

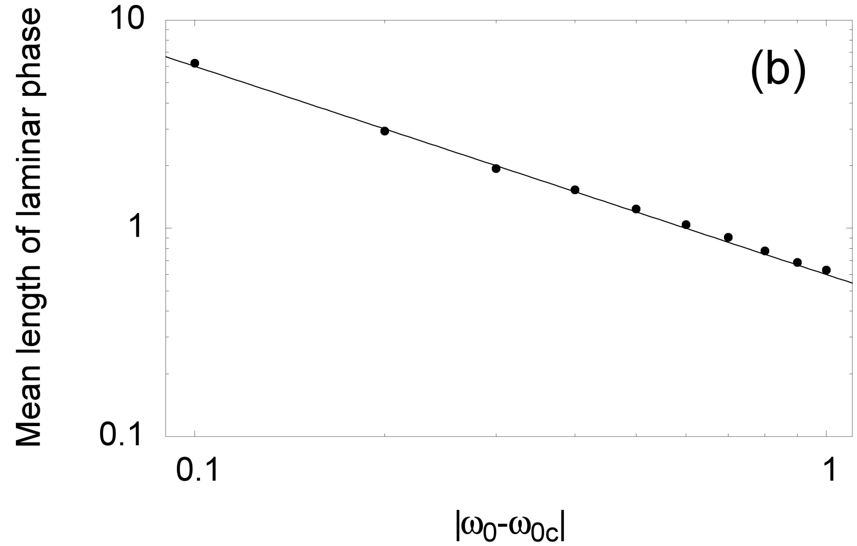
$$P(\tau) \sim \tau^{-3/2} \quad \text{for small} \quad \tau \to 0$$

(fully developed turbulence)

$$P(\tau) \sim e^{-2\varepsilon\tau} \quad \text{for large} \quad \tau \to \infty$$

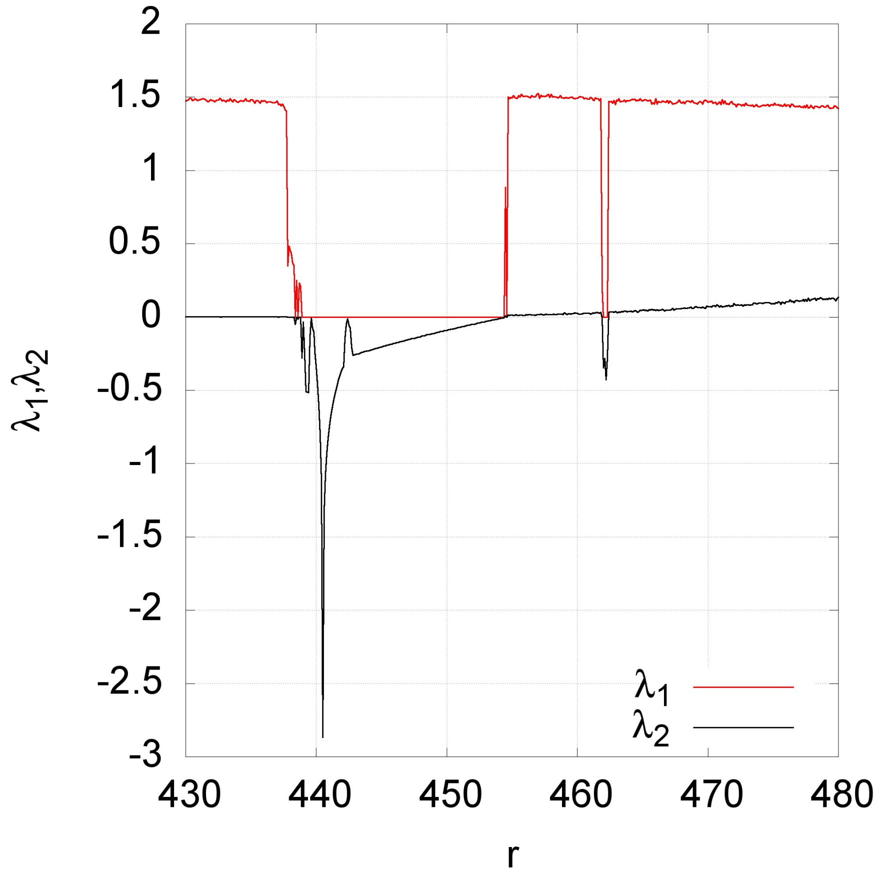
(self-organized criticality)





Scaling of the mean length of the laminar phase with control parameter $\varepsilon = |\omega_0 - \omega_{0c}|$, where ω_{0c} is a critical value at which intermittency appears: (a) for type I intermittency the dependence resulting from computations (circles) can be approximated by $\propto \varepsilon^{-1/2}$ function (solid line), (b) for type III intermittency $\propto \varepsilon^{-1}$, cf. (Macek and Strumik, 2014).

Hyperchaotic Convection



Dependence of the two largest Lyapunov exponents λ_1 (red line) and λ_2 (black line), $\lambda_1 > \lambda_2$, on the parameter r of the system for fixed values of the other parameters: $\omega_0 = 5.95$, $\sigma_{\mathrm{m}} = 0.1$, $\sigma = 10$, and $b = 8/3$. A transition to hyperchaotic dynamics is observed, when the second Lyapunov exponent λ_2 becomes positive for $r \geq 454.7$, taken from (Macek and Strumik, 2014).

Possible Applications of the Model

- liquid interiors of the Earth's core (the geodynamo model),
- interiors of the Sun and stars, including massive stars with heavy elements (*Brite* experiment),
- solar sunspots and coronal holes, granulation;
- the flow in the magnetosphere and heliosphere, and even in interstellar and intergalactic media;
- magneto-confined plasmas in tokamaks;
- nanodevices and microchannels in nanotechnolgy.

Conclusions

- We have proposed a new low-dimensional model describing self-consistently convective transport of the magnetic field applied along a horizontal layer of a viscous fluid. In addition to the usual three-dimensional Lorenz model a new variable describes the profile of the induced magnetic field. Nonperiodic oscillations are influenced by anisotropic magnetic forces resulting not only in an additional viscosity but also substantially modifying nonlinear forcing of the system.

- This four-dimensional dynamical system exhibits quite unusual features depending on the control parameters of the model. More specifically, by increasing an initial temperature difference and magnetic field strength one can switch-on and -off between nonperiodic (chaotic), periodic (limit cycle), and equilibrium (fixed point) asymptotic solutions.

- In addition, because of fine structure illustrated in the space of both control parameters, the influence of the induced magnetic field of the properties of the fluid could be much more intricate than a simple stabilizing effect predicted by simplified analysis of influence of the magnetic field on convective motion discussed in textbooks, see, e.g. (Cowling, 1976). This is interesting because there are physical situations where even weak field may have strong destabilizing effect.

- In particular, still in a chaotic regime but near the border with periodic solutions, in addition to previously identified type III intermittency, we have also observed type I intermittent behavior of the system that could provide new mechanisms of release of kinetic and magnetic energy bursts. It is worth noting that the observed sudden transitions from regular to irregular behavior only mimic stochastic forces, but in fact they result from nonlinearity, i.e., they are due to the disappearance of the fixed points of the dynamical system or owing to change in their their stability.

- In this way, we have identified here a basic mechanism of intermittent release of energy bursts, $\nu|\mathbf{v}|^2 + \eta|\mathbf{B}|^2/(\mu_o\rho)$, which is frequently observed in space and laboratory plasmas.

- It is important to note that besides the chaotic behavior well known for the Lorenz model with unmagnetized fluid, we have also identified here for the first time a hyperchaotic dynamics, with two positive Lyapunov exponents appearing for some value of the intensity of the applied magnetic field (Macek and Strumik, 2014). In this context the new hyperchaotic system characterized by both types I and III of intermittent energy release may provide an approximate description of irregular convective dynamical processes observed often in various plasmas in both space and laboratory.

- Hence we hope that our simple but still a more general nonlinear model could shed light on the nature of hydromagnetic turbulent convection, helping to identify chaotic and intermittent behavior in various environments.

- We propose this model as a useful tool for analysis of intermittent behavior of various environments, including convection in planets and stars.

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