# Introduction to Fractals, Chaos, Intermittency, and Multifractal Turbulence

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# Objective

The aim of the tutorial is to give students an introduction to the new developments in nonlinear dynamics and fractals. Based on intuition rather than mathematical proofs, emphasis will be on the basic concepts of fractals, stability, nonlinear dynamics, leading to strange attractors, deterministic chaos, bifurcations, and intermittency. The specific exercises will also include applications to intermittent turbulence in various environments. On successful completion of this brief tutorial, students should understand and apply the fractal models to real systems and be able to evaluate the importance of nonlinearity and multifractality, with possible applications to physics, astrophysics and space physics, and possibly chemistry, biology, and even economy.

# **Plan of the Tutorial**

- 1. Introduction
  - Dynamical and Geometrical View of the World
  - Fractals
  - Stability of Linear Systems
- 2. Nonlinear Dynamics
  - Attracting and Stable Fixed Points
  - Nonlinear Systems: Pendulum
- 3. Fractals and Chaos
  - Strange Attractors and Deterministic Chaos
  - Bifurcations

### 4. Multifractals

- Intermittent Turbulence
- Weighted Two-Scale Cantors Set
- Multifractals Analysis of Turbulence
- 5. Applications and Conclusions
  - Importance of Being Nonlinear
  - Importance of Multifractality

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff Kolmogorov Arnol'd Moser	Complex behavior in Hamiltonian mechanics
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications



# **Fractals**

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole.

Fractals are generally *self-similar* and independent of scale (fractal dimension).



If  $N_n$  is the number of elements of size  $r_n$  needed to cover a set (*C* is a constant) is:

$$N_n = \frac{C}{r_n^D},\tag{1}$$

then in case of self-similar sets:  $N_{n+1} = C/(r_{n+1})^D$ , and hence the fractal similarity dimension *D* is

$$D = \ln(N_{n+1}/N_n) / \ln(r_n/r_{n+1}).$$
 (2)

- Cantor set  $D = \ln 2 / \ln 3$
- Koch curve  $D = \ln 4 / \ln 3$
- Sierpinski carpet  $D = \ln 8 / \ln 3$
- Mengor sponge  $D = \ln 20 / \ln 3$
- Fractal cube  $D = \ln 6 / \ln 2$



## **Stability of Linear Systems**

**Two-Dimensional System** 

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solutions

$$x(t) = x_o e^{at}$$
$$y(t) = y_o e^{-t}$$





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## **Attracting and Stable Fixed Points**

We consider a fixed point  $x^*$  of a system  $\dot{x} = F(x)$ , where  $F(x^*) = 0$ .

We say that  $\mathbf{x}^*$  is *attracting* if there is a  $\delta > 0$  such that  $\lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}^*$ whenever  $|\mathbf{x}(0) - \mathbf{x}^*|| < \delta$ : any trajectory that starts within a distance  $\delta$  of  $\mathbf{x}^*$  is guaranteed to converge to  $\mathbf{x}^*$ .

A fixed point  $\mathbf{x}^*$  is *Lyapunov stable* if for each  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$  whenever  $t \ge 0$  and  $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$  : all trajectories that start within  $\delta$  of  $\mathbf{x}^*$  remain within  $\varepsilon$  of  $\mathbf{x}^*$  for all positive time.



## **Nonlinear Systems: Pendulum**

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# Attractors

An **ATTRACTOR** is a *closed* set *A* with the properties:

- 1. *A* is an INVARIANT SET: any trajectory  $\mathbf{x}(t)$  that start in *A* stays in *A* for ALL time *t*.
- 2. A ATTRACTS AN OPEN SET OF INITIAL CONDITIONS: there is an open set U containing  $A (\subset U)$  such that if  $\mathbf{x}(0) \in U$ , then the distance from  $\mathbf{x}(t)$  to A tends to zero as  $t \to \infty$ .
- 3. *A* is MINIMAL:

there is NO proper subset of A that satisfies conditions 1 and 2.

**STRANGE ATTRACTOR** is an attracting set that is a fractal: has zero measure in the embedding phase space and has FRACTAL dimension. Trajectories within a strange attractor appear to skip around randomly.

Dynamics on **CHAOTIC ATTRACTOR** exhibits sensitive (exponential) dependence on initial conditions (the 'butterfly' effect).

# **Deterministic Chaos**

CHAOS ( $\chi \alpha \circ \varsigma$ ) is

- NON-PERIODIC long-term behavior
- in a DETERMINISTIC system
- that exhibits SENSITIVITY TO INITIAL CONDITIONS.

We say that a bounded solution  $\mathbf{x}(t)$  of a given dynamical system is SENSITIVE TO INITIAL CONDITIONS if there is a finite fixed distance r > 0such that for any neighborhood  $||\Delta \mathbf{x}(0)|| < \delta$ , where  $\delta > 0$ , there exists (at least one) other solution  $\mathbf{x}(t) + \Delta \mathbf{x}(t)$  for which for some time  $t \ge 0$  we have  $||\Delta \mathbf{x}(t)|| \ge r$ .

There is a fixed distance r such that no matter how precisely one specify an initial state there is a nearby state (at least one) that gets a distance raway.

Given  $\mathbf{x}(\mathbf{t}) = \{x_1(t), \dots, x_n(t)\}$  any positive finite value of Lyapunov exponents  $\lambda_k = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\Delta x_k(t)}{\Delta x_k(0)} \right|$ , where  $k = 1, \dots, n$ , implies chaos.

	ATTRACTOR	LYAPUNOV Exponent Spectrum
c	LIMIT CYCLE	(0,-,-)
•	POINT	(-,-,-)
E	STRANGE ATTRACTOR	(+,0,-)

## **Types of Bifurcations**

Figure 2.15 Generic bifurcations of differentiable onedimensional maps. The system parameter r increases toward the right. The vertical scale represents the value of the map variable. Dashed lines represent unstable orbits. Solid lines represent stable orbits.



## **Bifurcation Diagram for the Logistic Map**



# Intermittency

In dynamical systems theory: occurrence of a signal that alternates randomly between long periods of regular behavior and relatively short irregular bursts. In other words, motion in intermittent dynamical system is nearly periodic with occasional irregular bursts.



Pomeau & Manneville, 1980

## **Intermittent Behavior**





### **Bifurcation and Intermittency**





## **Fractals and Multifractals**

A **fractal** is a rough or fragmented geometrical object that can be subdivided in parts, each of which is (at least approximately) a reduced-size copy of the whole. Fractals are generally *self-similar* and independent of scale (fractal dimension).

A **multifractal** is a set of intertwined fractals. Self-similarity of multifractals is scale dependent (spectrum of dimensions). A deviation from a strict self-similarity is also called **intermittency**.



Two-scale **Cantor** set.

### Fractal

A measure (volume) V of a set as a function of size l

 $V(l) \sim l^{D_{\rm F}}$ 

The number of elements of size *l* needed to cover the set

 $N(l) \sim l^{-D_{\rm F}}$ 

The fractal dimension

$$D_{\rm F} = \lim_{l \to 0} \frac{\ln N(l)}{\ln 1/l}$$

### **Multifractal**

A (probability) measure versus singularity strength,  $\alpha$ 

 $p_i(l) \propto l^{\alpha_i}$ 

The number of elements in a small range from  $\alpha$  to  $\alpha$  +  $\mathrm{d}\alpha$ 

 $N_l(\alpha) \sim l^{-f(\alpha)}$ 

The multifractal singularity spectrum

$$f(\alpha) = \lim_{\epsilon \to 0} \lim_{l \to 0} \frac{\ln[N_l(\alpha + \epsilon) - N_l(\alpha - \epsilon)]}{\ln 1/l}$$

#### The generalized dimension

$$D_{q} = \frac{1}{q - 1} \lim_{l \to 0} \frac{\ln \sum_{i=1}^{N} (p_{i})^{q}}{\ln l}$$

## **Multifractal Characteristics**



**Fig. 1.** (a) The generalized dimensions  $D_q$  as a function of any real q,  $-\infty < q < \infty$ , and (b) the singularity multifractal spectrum  $f(\alpha)$  versus the singularity strength  $\alpha$  with some general properties: (1) the maximum value of  $f(\alpha)$  is  $D_0$ ; (2)  $f(D_1) = D_1$ ; and (3) the line joining the origin to the point on the  $f(\alpha)$  curve where  $\alpha = D_1$  is tangent to the curve (Ott *et al.*, 1994).

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### **Generalized Scaling Property**

The generalized dimensions are important characteristics of *complex* dynamical systems; they quantify multifractality of a given system (Ott, 1993).

Using  $(\sum p_i^q \equiv \langle p_i^{q-1} \rangle_{av})$  a generalized average probability measure of cascading eddies

$$\bar{u}(q,l) \equiv \sqrt[q-1]{\langle (p_i)^{q-1} \rangle_{\rm av}}$$
(3)

we can identify  $D_q$  as scaling of the measure with size l

$$\bar{\mu}(q,l) \propto l^{D_q} \tag{4}$$

Hence, the slopes of the logarithm of  $\bar{\mu}(q,l)$  of Eq. (4) versus  $\log l$  (normalized) provides

$$D_q = \lim_{l \to 0} \frac{\log \bar{\mu}(q, l)}{\log l} \tag{5}$$

#### **Measures and Multifractality**

Similarly, we define a one-parameter q family of (normalized) generalized pseudoprobability measures (Chhabra and Jensen, 1989; Chhabra *et al.*, 1989)

$$\mu_i(q,l) \equiv \frac{p_i^q(l)}{\sum_{i=1}^N p_i^q(l)} \tag{6}$$

Now, with an associated fractal dimension index  $f_i(q,l) \equiv \log \mu_i(q,l) / \log l$  for a given q the multifractal spectrum of dimensions is defined directly as the averages taken with respect to the measure  $\mu(q,l)$  in Eq. (6) denoted by  $\langle \ldots \rangle$ 

$$f(q) \equiv \lim_{l \to 0} \sum_{i=1}^{N} \mu_i(q,l) f_i(q,l) = \lim_{l \to 0} \frac{\langle \log \mu_i(q,l) \rangle}{\log(l)}$$
(7)

and the corresponding average value of the singularity strength is given by (Chhabra and Jensen, 1987)

$$\alpha(q) \equiv \lim_{l \to 0} \sum_{i=1}^{N} \mu_i(q, l) \; \alpha_i(l) = \lim_{l \to 0} \; \frac{\langle \log p_i(l) \rangle}{\log(l)}. \tag{8}$$

## **Turbulence Cascade**



**Fig. 1.** Schematics of binomial multiplicative processes of cascading eddies. A large eddy of size *L* is divided into two smaller *not necessarily equal* pieces of size  $l_1$  and  $l_2$ . Both pieces may have different probability measures, as indicated by the different shading. At the *n*-th stage we have  $2^n$  various eddies. The processes continue until the Kolmogorov scale is reached (Meneveau and Sreenivasan, 1991; Macek, 2012).

In the case of turbulence cascade these generalized measures are related to inhomogeneity with which the energy is distributed between different eddies (Meneveau and Sreenivasan, 1991). In this way they provide information about dynamics of multiplicative process of cascading eddies. Here high positive values of q > 1 emphasize regions of intense fluctuations larger than the average, while negative values of q accentuate fluctuations lower than the average (cf. Burlaga 1995).

## **Methods of Data Analysis**

### **Structure Functions Scaling**

In the inertial range ( $\eta \ll l \ll L$ ) the averaged standard *q*th order (q > 0) structure function is scaling with a scaling exponent  $\xi(q)$  as

$$S_u^q(l) = \langle |u(x+l) - u(x)|^q \rangle_{\rm av} \propto l^{\xi(q)}$$
(9)

where u(x) and u(x+l) are velocity components parallel to the longitudinal direction separated from a position *x* by a distance *l*.

The existence of an inertial range for the experimental data is discussed by Horbury *et al.* (1997), Carbone (1994), and Szczepaniak and Macek (2008).

### **Energy Transfer Rate and Probability Measures**

$$\varepsilon(x,l) \sim \frac{|u(x+l) - u(x)|^3}{l},\tag{10}$$

To each *i*th eddy of size *l* in turbulence cascade ( $i = 1, ..., N = 2^n$ ) we associate a probability measure

$$p(x_i, l) \equiv \frac{\varepsilon(x_i, l)}{\sum_{i=1}^N \varepsilon(x_i, l)} = p_i(l).$$
(11)

This quantity can roughly be interpreted as a probability that the energy is transferred to an eddy of size  $l = v_{sw}t$ .

As usual the time-lags can be interpreted as longitudinal separations,  $x = v_{sw}t$  (Taylor's hypothesis).

#### **Magnetic Field Strength Fluctuations and Generalized Measures**

Given the normalized time series  $B(t_i)$ , where  $i = 1, ..., N = 2^n$  (we take n = 8), to each interval of temporal scale  $\Delta t$  (using  $\Delta t = 2^k$ , with k = 0, 1, ..., n) we associate some probability measure

$$p(x_j, l) \equiv \frac{1}{N} \sum_{i=1+(j-1)\Delta t}^{j\Delta t} B(t_i) = p_j(l),$$
(12)

where  $j = 2^{n-k}$ , i.e., calculated by using the successive (daily) average values  $\langle B(t_i, \Delta t) \rangle$  of  $B(t_i)$  between  $t_i$  and  $t_i + \Delta t$ . At a position  $v_{sw}$ , at time t, where  $v_{sw}$  is the average solar wind speed, this quantity can be interpreted as a probability that the magnetic flux is transferred to a segment of a spatial scale  $l = v_{sw}\Delta t$  (Taylor's hypothesis).

The average value of the qth moment of the magnetic field strength B should scale as

$$\langle B^q(l) \rangle \sim l^{\gamma(q)},$$
 (13)

with the exponent  $\gamma(q) = (q-1)(D_q-1)$  as shown by Burlaga *et al.* (1995).

## **Mutifractal Models for Turbulence**



Fig. 1. Generalized two-scale Cantor set model for turbulence (Macek, 2007).

## **Degree of Multifractality and Asymmetry**

The difference of the maximum and minimum dimension (the least dense and most dense points in the phase space) is given, e.g., by Macek (2006, 2007)

$$\Delta \equiv \alpha_{\max} - \alpha_{\min} = D_{-\infty} - D_{\infty} = \left| \frac{\log(1-p)}{\log l_2} - \frac{\log(p)}{\log l_1} \right|.$$
(14)

In the limit  $p \rightarrow 0$  this difference rises to infinity (degree of multifractality).

The degree of multifractality  $\Delta$  is simply related to the deviation from a simple selfsimilarity. That is why  $\Delta$  is also a measure of intermittency, which is in contrast to selfsimilarity (Frisch, 1995, chapter 8).

Using the value of the strength of singularity  $\alpha_0$  at which the singularity spectrum has its maximum  $f(\alpha_0) = 1$  we define a measure of asymmetry by

$$A \equiv \frac{\alpha_0 - \alpha_{\min}}{\alpha_{\max} - \alpha_0}.$$
 (15)

## **Importance of Multifractality**

Starting from seminal works of Kolmogorov (1941) and Kraichnan (1965) many authors have attempted to recover the observed scaling exponents, using multifractal phenomenological models of turbulence describing distribution of the energy flux between cascading eddies at various scales (Meneveau and Sreenivasan, 1987, Carbone, 1993, Frisch, 1995).

The concept of multiscale multifractality is of great importance for space plasmas because it allows us to look at intermittent turbulence in the solar wind. In particular, the multifractal spectrum has been investigated and using Helios (plasma) data in the inner heliosphere (e.g., Marsch *et al.*), Voyager (magnetic field fluctuations) data in the outer heliosphere (e.g., Burlaga, 1991, 1995, 2001; Burlaga and Ness, 2010, 2014; Burlaga *et al.*, 1993, 2003, 2006, 2007, 2013, 2015).

We have first analysed the multifractal spectrum directly on the solar wind attractor and have shown that it is consistent with that for the multifractal measure of a two-scale weighted Cantor set (Macek, 2007).

Next, we have analysed solar wind plasma with frozen-in magnetic field based on data acquired during space missions onboard various spacecraft, such as Helios (Macek and Szczepaniak, 2008), Advanced Composition Explorer (Szczepaniak and Macek, 2008), Ulysses (Wawrzaszek and Macek, 2010), and Voyager (Macek and Wawrzaszek, 2009), exploring different regions of the heliosphere during solar minimum and maximum.

Recently, we have looked at the fluctuations of the interplanetary magnetic fields observed by both Voyager 1 and 2 spacecraft in the outer heliosphere and the heliosheath (Macek *et al.*, 2011, 2012), i.e., after crossing the heliospheric termination heliospheric shock at 94 and 84 AU (in 2004 and 2007, respectively), and finally at 122 AU (2012) the heliopause (Macek *et al.*, 2014), which is the last boundary separating the heliospheric plasma from the local interstellar plasma.

# Conclusions

- Fractal structure can describe complex shapes in the real word.
- Nonlinear systems exhibit complex phenomena, including bifurcation, intermittency, and chaos.
- Strange chaotic attractors have fractal structure and are sensitive to initial conditions.
- Within the complex dynamics of the fluctuating intermittent parameters of turbulent media there is a detectable, hidden order described by a generalized Cantor set that exhibits a multifractal structure.

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