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Influence of dynamical noise on time series generated by nonlinear maps

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Abstract

We consider periodic and chaotic dynamics of discrete nonlinear maps in the presence of dynamical noise. We show that dynamical noise corrupting dynamics of a nonlinear map may be considered as a measurement "pseudonoise" with the distribution determined by the Jacobian of the map. The formula for the distribution of the measurement "pseudonoise" for one-dimensional quadratic maps has also been obtained in an explicit form. We expect that our results apply to an arbitrary distribution of low-level dynamical noise and hope that these results could help to find a universal method of discriminating dynamical from measurement noise.

Keywords: Chaotic systems; Noise; Time series analysis

1. Introduction

Developing a model of a real physical dynamical system and comparing the results of the modelling with a measured signal, we always observe departures of the model from the real dynamics. These departures are caused by two reasons. First, during a measurement process some contaminations can be superimposed on the real signal. We assume that these contaminations do not change dynamics of the system, but rather result from imperfections of the measurement process. Therefore, this kind of contamination is called measurement noise. However, noise can enter dynamics in a more complex way. Namely, the applied deterministic model of the dynamics could be inexact and the system could rather evolve according to a rule consisting of combined deterministic and stochastic components. This kind of contaminations is called dynamical noise. Dynamical noise usually represents an intrinsic random process being superimposed on deterministic dynamics. Such a situation may appear, e.g. when we have a real physical system, for which a low-dimensional model is a quite good approximation, but random (i.e. high-dimensional) forces perturb the dynamics causing some deviations from pure low-dimensional evolution. This effect may occur, e.g. in chaotic flow of a fluid, where low-dimensional temporal evolution may be corrupted by effects related to the lack of perfect spatial coherence in the system.

Measurement noise is a relatively simple case to work with, even for such complex time series as those generated by chaotic dynamical systems. It is possible to estimate noise level for measurement noise (see, e.g. the algorithms described in Refs. [1–4] and references therein) and perform considerable noise reduction [5–7]. On the contrary, dynamical noise is a much more difficult case, mainly because this kind of contamination is strongly involved in nonlinear dynamics of the systems. It is still possible to estimate noise level for chaotic time series contaminated by dynamical noise (see Refs. [4,8]), but the question of noise reduction is rather poorly understood [9]. The issue of influence of dynamical noise on dynamics of discrete maps or ordinary differential equations has been investigated in Refs. [10–14]. In particular, an interesting technique of reconstructing the Langevin equation from data has been also proposed [15,16].

In this paper we focus on the question of dynamical noise representing uncorrelated random forces affecting low-dimensional dynamics given by deterministic maps. Our

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goal is to provide a theoretical basis for analysis of the influence of dynamical noise on time series generated by the maps. This paper is an extension of Ref. [8], where a method of estimation of noise level and discrimination of dynamical from measurement noise has been proposed for a time series generated by chaotic discrete maps. The method is based on the observation that dynamical noise, when considered as measurement "pseudonoise", exhibits departures from Gaussian behaviour toward the Cauchy distribution. This effect leads to statistically significant differences, e.g. in scaling properties of the correlation entropy, which can be used for discriminating measurement from dynamical noise for chaotic time series. In this paper we try to explain the mechanism of generation of the specific distribution of the measurement "pseudonoise" for time series originated from chaotic maps corrupted by dynamical noise.

In Section 2 we present results of theoretical analysis of the influence of dynamical noise on one-dimensional maps. In Section 3 the results are generalized to the case of d-dimensional maps. Our considerations are then verified in Section 4 using several examples of time series generated by both periodic and chaotic maps. Finally, we briefly summarize our results in Section 5.

2. Influence of dynamical noise on one-dimensional maps

In the literature devoted to the problem of dynamical noise corrupting dynamics of discrete maps of the form

$$x_{n+1} = f(x_n), \tag{1}$$

we can find the following two models of the influence of dynamical noise on dynamics of the maps:

$$y_{n+1} = f(y_n) + \eta_n \tag{2}$$

and

$$z_{n+1} = f(z_n + \eta_n), \tag{3}$$

where η_n is a noise term corrupting the clean dynamics described by Eq. (1). We assume here that from the point of view of analysis of time series generated by discrete maps, dynamical noise may be considered as the measurement "pseudonoise". Let us now find the difference between the models of Eqs. (2) and (3) as seen from this point of view.

Let us write step-by-step iterations of the noisy map of Eq. (2), assuming that dynamical noise results in the measurement "pseudonoise" affecting time series generated by the maps:

$$\begin{aligned} &x_1 + \eta_0 = f(x_0) + \eta_0, \\ &x_2 + \epsilon_2 + \eta_1 = f(x_1 + \eta_0) + \eta_1, \\ &x_3 + \epsilon_3 + \eta_2 = f(x_2 + \epsilon_2 + \eta_1) + \eta_2, \\ &\vdots \\ &x_{n+1} + \epsilon_{n+1} + \eta_n = f(x_n + \epsilon_n + \eta_{n-1}) + \eta_n. \end{aligned} \tag{4}$$

Here x_n is a time series generated by the clean map (1), η_n is a noise term corrupting the map, and ϵ_{n+1} is an additional noise term resulting from the fact that in the case of noisy map the function f operates on the perturbed state of the system. Therefore, for Eq. (2) we have ($n \geq 2$)

$$x_{n+1} + \epsilon_{n+1} + \eta_n = f(x_n + \epsilon_n + \eta_{n-1}) + \eta_n \tag{5}$$

and similarly for Eq. (3)

$$x_{n+1} + \epsilon_{n+1} = f(x_n + \epsilon_n + \eta_n). \tag{6}$$

Eqs. (5) and (6) remain valid even when the function f is nonlinear, because we do not impose any specific form on ϵ_{n+1}. Assuming that the noise level (standard deviation of η_n) is small, one may use the Taylor expansion limited to the derivative of the first order to obtain

$$x_{n+1} + \epsilon_{n+1} + \eta_n = f(x_n) + f'(x_n)(\epsilon_n + \eta_{n-1}) + \eta_n \tag{7}$$

and

$$x_{n+1} + \epsilon_{n+1} = f(x_n) + f'(x_n)(\epsilon_n + \eta_n) \tag{8}$$

as appropriate approximations of Eqs. (5) and (6), respectively. Now, we can expect that it is possible to extract a stochastic process (denoted by $\hat{\epsilon}_n$) satisfying the following condition

$$\hat{\epsilon}_{n+1} = f'(x_n)(\hat{\epsilon}_n + \eta_n), \tag{9}$$

which should reproduce statistical properties of the stochastic process ϵ_n involved in Eqs. (7) and (8).

The process ϵ_n determines the behaviour of the measurement "pseudonoise" for a given map. Admittedly, direct investigations of this process are not possible when it is involved in chaotic dynamics of a map, because it requires a good method of finding the clean trajectory for chaotic dynamics corrupted by dynamical noise. However, having the stochastic process isolated in the form given by Eq. (9), we can easily examine its statistical properties.

Eqs. (7) and (8) reveal a basic difference between noisy time series generated by the models of Eqs. (2) and (3). Namely, in the case of the model of Eq. (2), the distribution of measurement "pseudonoise" is determined as the distribution of random variable being the sum of the variables ϵ_{n+1} and η_n. Whereas for the model of Eq. (3), the distribution of measurement "pseudonoise" is determined as the distribution of only one random variable ϵ_{n+1}. In both cases, properties of the random variable ϵ_n seem to be crucial for understanding the behaviour of measurement "pseudonoise" contaminating time series generated by nonlinear maps. We expect that statistical properties of ϵ_n can be revealed by analysis of the isolated stochastic process $\hat{\epsilon}_n$ described by Eq. (9).

Some comments are necessary here. The stochastic process of Eq. (9) naturally appears as involved in dynamical noise contaminated dynamics of a discrete map (see Eq. (5) or (6)). But when we isolate the process as it is shown in Eq. (9), it has such a character that a sequence of values of $|f'(x_n)|$ larger then one can cause infinite growth of $\hat{\epsilon}_n$ with increasing n. Therefore, to avoid such a situation, iterating directly the isolated process described by Eq. (9), we need to use rescaled values of $f'(x_n)$, so that the maximal value of $s|f'(x_n)|$ will be less or equal to one (s is here a scaling factor).

To examine the distribution of the measurement "pseudonoise", we first consider the process of Eq. (9) for the case of the map of period one, i.e. for steady states of these maps. Because in this situation for the clean dynamics we have $x^* = f(x^*)$, then for the noisy dynamics the points x_n are spread over a neighbourhood of x^*. Therefore, one may expect that in this case the value of $f'(x_n)$ is approximately constant, i.e. $f'(x_n) \approx f'(x^*)$. In this situation Eq. (9) can be reduced to $\hat{\epsilon}_{n+1} = c\,(\hat{\epsilon}_n + \eta_n)$, where $c = f'(x^*)$ is a constant. Therefore, using the central limit theorem we can conclude that in this case $\hat{\epsilon}_n$ follows the Gaussian distribution

$$\frac{1}{\sigma_g\sqrt{2\pi}} \exp\left[-\frac{\epsilon^2}{2\sigma_g^2}\right], \tag{10}$$

irrespective of the distribution of η_n. The standard deviation of the Gaussian distribution is proportional to $|c|$. We use here two assumptions. First, we assume that $f'(x)$ is smooth and does not vary too much in the neighbourhood of x^*. The second assumption is that the noise level is relatively small and η_n is limited to a finite range of values. This is a necessary condition, because otherwise for a large value of η_n a perturbed state of the dynamical system can jump out of the basin of attraction of the point x^*, changing completely a given trajectory. Hence a formal question appears, because if η_n is, e.g. Gaussian distributed then there is no limit on the values of η_n and every trajectory of infinite length ultimately jumps out of the basin of attraction of the point x^*. Practically, this problem does not arise as far as noise level is relatively small and we deal with time series of a finite length.

In the case of a more general situation of a map of period p corrupted by dynamical noise, its trajectory goes in cycles through neighbourhoods of points x^{*i} ($i = 1, \ldots, p$) and at every i-th point of the cycle the value of $c^{*i} = f'(x^{*i})$ is different in general. The time series generated by such a map can be considered as comprised of p groups of time series. Time series representing i-th group have properties similar to the case of the map of period one, i.e. the distribution of $\hat{\epsilon}_n$ is determined by a reduced version of Eq. (9), namely $\hat{\epsilon}_{n+1} = c^{*i}(\hat{\epsilon}_n + \eta_n)$, which means that for a given i, $\hat{\epsilon}_n$ is Gaussian distributed (irrespective of the distribution of η_n, as it was argued earlier for the case of the map of period one) with the standard deviation proportional to $|c^{*i}|$. Therefore, for time series generated by periodic maps of period p contaminated by dynamical noise, $\hat{\epsilon}_n$ follows the distribution

$$g_p(\hat{\epsilon}) = \sum_{i=1}^{p} g_p^{*i}(\hat{\epsilon}) = \sum_{i=1}^{p} \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left[-\frac{\hat{\epsilon}^2}{2\sigma_i^2}\right], \tag{11}$$

where $\sigma_i = |c^{*i}|\sigma_\eta$, and σ_η is the standard deviation of η_n. Obviously, also in the case of the map of period p, the conclusions should be valid providing that noise level is small and the function $f'(x)$ is slowly varying in neighbourhoods of the points x^{*i}.

When computing some statistical measures, chaotic maps can be considered as periodic maps of period $p \to \infty$. Therefore, the conclusions that have been obtained for the case of periodic map of period p contaminated by dynamical noise should, in principle, remain valid also for chaotic maps. However, in the case of chaotic maps, the sum in Eq. (11) must be replaced by the integral

$$g_c(\hat{\epsilon}) = \int_{x_{\min}}^{x_{\max}} \rho(x) \frac{1}{\sigma(x)\sqrt{2\pi}} \exp\left[-\frac{\hat{\epsilon}^2}{2\sigma^2(x)}\right] \mathrm{d}x. \tag{12}$$

Here $\rho(x)$ is the invariant density and $\sigma(x) = |f'(x)|\sigma_\eta$, where σ_η is the standard deviation of η_n, and $x_{\min}$ and $x_{\max}$ are correspondingly the minimal and maximal values of x_n determining the range of variability of x_n. Obviously, we assume here that the dynamical noise level σ_η is small. Therefore, we can also expect that dynamical noise contaminating a discrete map does not change substantially the invariant density of the clean map. In fact, the influence of dynamical noise results basically in smoothing the invariant density of the clean map. Hence we can use the invariant density of the clean map in Eq. (12). Furthermore, because of the averaging character of Eq. (12), we expect that often to simplify the calculations, $\rho(x) = 1/(x_{\max} - x_{\min}) = \text{const}$ could be a reasonable approximation, provided that $\rho(x)$ is not correlated (positively or negatively) with $\sigma(x) = |f'(x)|\sigma_\eta$. However, it may happen sometimes, that within the interval $(x_{\min}, x_{\max})$ the invariant density for the clean map has subintervals where $\rho(x) = 0$. In this situation the approximation $\rho(x) = 1/(x_{\max} - x_{\min})$ is obviously not appropriate for entire interval $(x_{\min}, x_{\max})$. In Fig. 1(d) we present a comparison of the theoretical $g_c(\hat{\epsilon})$ (computed using the approximation $\rho(x) = \text{const}$) with the distribution obtained by direct iterations of Eq. (9). One can see that the errors introduced by the approximation $\rho(x) = \text{const}$ are not large.

For quadratic maps we have $f'(x) = ax + b$. If we additionally assume that for simplicity reasons we can use $\rho(x) = 1/(x_{\max} - x_{\min})$, then it is possible to find an explicit formula for the distribution of $\hat{\epsilon}_n$. Namely, for $x_{\min} < -b/a < x_{\max}$ we have

$$g_c(\hat{\epsilon}) = \pm \frac{E_1\left(\frac{\hat{\epsilon}^2}{2\sigma^2(ax_{\min}+b)^2}\right) + E_1\left(\frac{\hat{\epsilon}^2}{2\sigma^2(ax_{\max}+b)^2}\right)}{\sqrt{8\pi}\,\sigma a(x_{\max} - x_{\min})} \tag{13}$$

and otherwise

$$g_c(\hat{\epsilon}) = \pm \frac{E_1\left(\frac{\hat{\epsilon}^2}{2\sigma^2(ax_{\max}+b)^2}\right) - E_1\left(\frac{\hat{\epsilon}^2}{2\sigma^2(ax_{\min}+b)^2}\right)}{\sqrt{8\pi}\,\sigma a(x_{\max} - x_{\min})}, \tag{14}$$

where the signs of the expressions of Eqs. (13) and (14) are determined by the sign of a, and $E_1(x)$ is the exponential integral $E_m(x) = \int_1^\infty \exp(-xt)/t^m \mathrm{d}t$ taken for $m = 1$.

3. Generalization to d-dimensional maps

So far we have discussed only the influence of dynamical noise on one-dimensional discrete maps. Let us now consider a d-dimensional map

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) \tag{15}$$

corrupted by dynamical noise. Then Eq. (9) should be replaced by

$$\mathbf{e}_{n+1} = \mathbf{DF}(\mathbf{x}_n)(\mathbf{e}_n + \mathbf{h}_n). \tag{16}$$

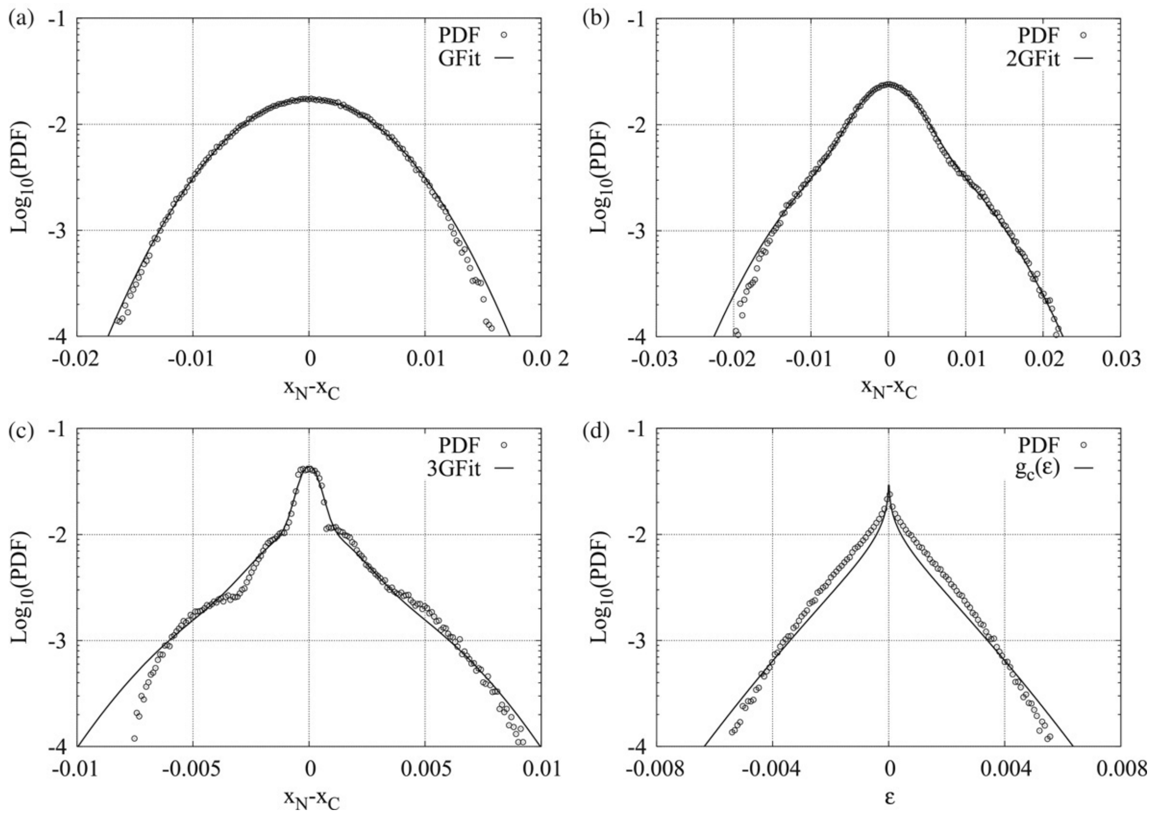

Fig. 1. The distributions (PDF) of the differences between noisy and clean time series for periodic cases of period (a) one ($r = 2.9$), (b) two ($r = 3.4$) and (c) three ($r = 3.835$) for the logistic map $x_{n+1} = rx_n(1 - x_n)$ are fitted by (a) Gaussian (GFit), (b) the sum of two Gaussians (2GFit), (c) the sum of three Gaussians (3GFit), correspondingly. In this numerical experiment we have used dynamical noise of uniform distribution in the range (a) $(-0.004, 0.004)$, (b) $(-0.0034, 0.0034)$, (c) $(-0.0006, 0.0006)$. Panel (d) shows the comparison of the function of Eq. (13) with the distribution obtained by direct iteration of rescaled version of Eq. (9) for a chaotic case ($r = 3.8$) of the logistic map.

Here the vector $\mathbf{e}_n$ corresponds to the scalar $\hat{\epsilon}_n$, the vector $\mathbf{h}_n$ corresponds to η_n, and the Jacobian of $\mathbf{F}(\mathbf{x})$ at $\mathbf{x}_n$ is denoted by $\mathbf{DF}(\mathbf{x}_n)$, where $(DF)_{ij} = \frac{\partial F_i}{\partial x_j}$. Because of the similarity of Eqs. (9) and (16) one may expect that the conclusions for the one-dimensional case should, in principle, be valid also for the d-dimensional case. Naturally, the formula (16) is a vector equation and for this reason the d-dimensional case can be somewhat more complicated.

For example, let us now examine Eq. (16) for the Hénon map [17]

$$x_{n+1} = 1 - ax_n^2 + y_n, \quad y_{n+1} = bx_n. \tag{17}$$

For this case we have

$$\begin{pmatrix} e^{x}_{n+1} \\ e^{y}_{n+1} \end{pmatrix} = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} e^{x}_{n} + h^{x}_{n} \\ e^{y}_{n} + h^{y}_{n} \end{pmatrix}. \tag{18}$$

Bearing in mind the conclusions obtained for the one-dimensional case, we can notice that e^x_n and e^y_n in Eq. (18) should in general have different distributions. Namely, we can infer from Eq. (18) that e^y_n should be Gaussian distributed with the standard deviation equal to $b\sigma_{h^x}$, where σ_{h^x} is the standard deviation of the x-component of $\mathbf{h}_n$. On the other hand, since the elements of the Jacobian in Eq. (18) determining the behavior of e^x_n are not all constant, therefore for e^x_n we expect some departures of the distribution from Gaussian. In fact, such a behaviour, i.e. non-Gaussian distribution for e^x_n and Gaussian for e^y_n, has been already reported in Ref. [8].

4. Examples

To see how our considerations work in practice, in this section we illustrate the theoretical considerations with examples of time series generated by model systems. In straightforward examination of the distribution of measurement “pseudonoise”, we subtract clean time series from noisy time series and analyse the distribution of the differences. Such a procedure can be easily applied for periodic systems, but application of such a method for chaotic (aperiodic) systems is very hard, because of serious difficulties with finding the clean trajectory for a given noisy trajectory. However, for chaotic systems one can examine the distribution indirectly, e.g. by examining the stochastic process of Eq. (9) (or Eq. (16) for the d-dimensional case) or the scaling properties of the correlation sum, dimension, or entropy.

The distributions of the differences between noisy time series x_N (corrupted by dynamical noise of uniform distribution) and clean time series x_C for the logistic map $x_{n+1} = rx_n(1-x_n)$ for periodic cases of period one ($r = 2.9$), two ($r = 3.4$) and three ($r = 3.835$) are shown in Fig. 1(a)–(c). For a comprehensive discussion on the dynamics of the logistic map see,

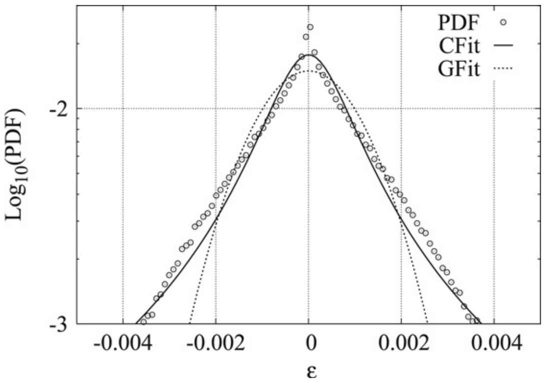

Fig. 2. The distribution of $\hat{\epsilon}_n$ for the logistic map for $r = 3.8$ obtained by iteration of the rescaled version of Eq. (9). The computed function (PDF) is fitted by Gaussian of Eq. (10) (GFit) and Cauchy function of Eq. (19) (CFit).

e.g. [18] or [19] and references therein. One can see that although the distribution of dynamical noise is uniform, the resulting probability distribution functions (PDF) are well approximated by Gaussian for the case of period one, or the sum of two and three Gaussians for the cases of periods two and three. In Fig. 1(a)–(c) one can see some deviations between probability distributions obtained numerically (points) and theoretically (lines). The deviations increase on average with increasing $x_N - x_C$, thus in our opinion, they are caused by violation of assumptions used in derivation of Eq. (9). In Fig. 1(d) we compare the function of Eq. (13) with the distribution obtained by direct iteration of rescaled version of Eq. (9) for a chaotic case ($r = 3.8$) of the logistic map. The results presented here for periodic and chaotic cases of the logistic map are in agreement with the conclusions obtained in Section 2, concerning the distribution of measurement "pseudonoise". In particular, we can see that in fact the stochastic process $\hat{\epsilon}_n$ of Eq. (9) properly describes statistical properties of the process ϵ_n involved in Eqs. (7) and (8).

The form of Eqs. (13) and (14) is quite complex, thus these formulae are rather not suitable for practical purposes. Admittedly, a function with exponentially decreasing flanks could be appropriate for the distribution shown in Fig. 1(d). However, for other cases, the core of the distribution of measurement "pseudonoise" is much smoother. In Ref. [8] the Cauchy function

$$\frac{1}{\sigma_c \pi \left(1 + \frac{\epsilon^2}{\sigma_c^2}\right)} \tag{19}$$

has been suggested as an approximation of the distribution of measurement "pseudonoise". In Fig. 2 we show the distribution of $\hat{\epsilon}_n$ computed for a chaotic case of the logistic map ($r = 3.8$) by direct iteration of Eq. (9). The obtained distribution of $\hat{\epsilon}_n$ is fitted by the Gaussian of Eq. (10) and Cauchy function of Eq. (19). One can see that the fit by Cauchy function is much better than the fit by Gaussian, especially in the range of $\hat{\epsilon}$ corresponding to the highest values of the probability distribution function; these values usually determine the behaviour of computed statistics.

Looking at Fig. 1(d) one can notice that the Cauchy function can be considered as a suitable approximation of the distribution obtained by the iteration of Eq. (9). Admittedly, the shape of the distribution of measurement "pseudonoise" predicted by Eq. (13) or (14) can change substantially depending on the parameters $a, b, \sigma, x_{\min}, x_{\max}$. In general, it may happen that the obtained shape will be completely different from Cauchy-like function. But we have verified that for the logistic map (in chaotic regime), the shape of the distribution is rather not very sensitive to the choice of the parameters. However, it is not possible yet to give a general answer to the question of whether or not the Cauchy function can be a good approximation of the distributions of measurement "pseudonoise" given by Eqs. (13) and (14); it would be rather difficult to speculate about universality of the Cauchy function approximation.

In Fig. 3 we show the distributions of e_n^x and e_n^y for a chaotic case of the Hénon map of Eq. (17) for $a = 1.4$, $b = 0.2$, computed by direct iteration of Eq. (18) and fitted by Gaussian and Cauchy functions. As can be seen, e_n^x is rather not Gaussian distributed and the fit by Cauchy function is better, whereas e_n^y is apparently well approximated by Gaussian. Such a behaviour, i.e. non-Gaussian distribution for e_n^x and Gaussian distribution for e_n^y has already been reported in Ref. [8], as resulting from the scaling properties of the correlation entropy for the Hénon map. Therefore, we can conclude that the results obtained in Section 3, concerning distributions of e_n^x and e_n^y for the Hénon map are correct.

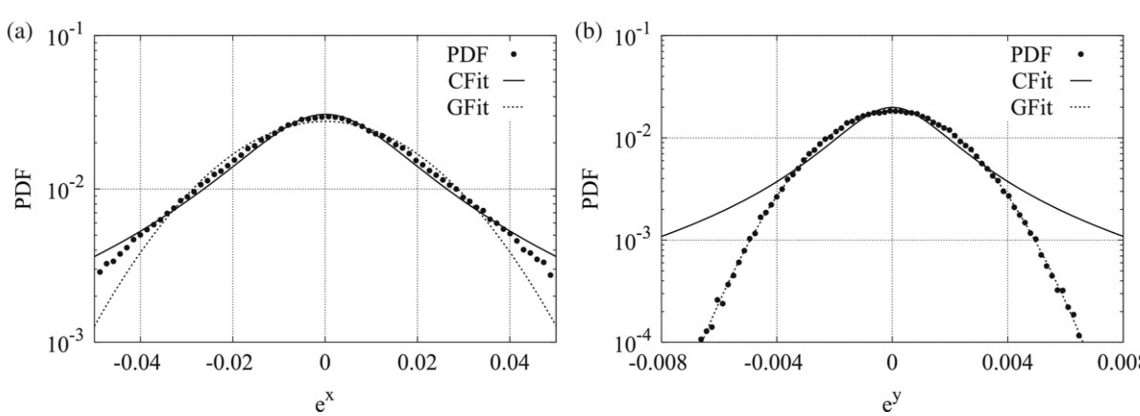

Fig. 3. Distributions of (a) e_n^x and (b) e_n^y obtained by the iteration of Eq. (18) for the Hénon map. The computed functions (PDF) are fitted by the Gaussian of Eq. (10) (GFit) and the Cauchy function of Eq. (19) (CFit).

5. Conclusions

We have shown that if dynamical noise corrupting a time series generated by a discrete map is considered as a measurement "pseudonoise", then the distribution of the "pseudonoise" is determined by the Jacobian of the map. When the Jacobian is constant, we may expect Gaussian distributed "pseudonoise" for one-dimensional maps, whereas for d-dimensional maps ($d \geq 2$), the distribution of the measurement "pseudonoise" is the distribution of the sum of at most d random variables, which themselves are Gaussian distributed (with different standard deviations in general). If the Jacobian is not constant, we may expect some departures of the "pseudonoise" distribution from the Gaussian. We have also obtained an explicit formula for the distribution of the measurement "pseudonoise" for one-dimensional quadratic maps. We have shown that the distribution of the measurement "pseudonoise" is independent of the distribution of dynamical noise, and thus we expect that the results obtained in this paper should apply to an arbitrary distribution of dynamical noise. The results obtained have been verified with several examples of time series generated by model systems. We hope that our results could help to find a universal method of distinguishing dynamical from measurement noise.

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