New Multifractal Analyses of the Solar Wind Turbulence : Rank-Ordered Multifractal Analysis and Generalized Two-scale Weighted Cantor Set Model

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Abstract. Since the work of Burlaga in 1991, various multifractal analyses have been performed to better characterize the intermittency of the fluctuations in the turbulent solar wind. These analyses were based either on the classical structure functions or on the partition function. Recently, two new multifractal analyses have been proposed to better characterize intermittent fluctuations: on one hand, Chang & Wu (2008) proposed a rank-ordered multifractal analysis based on range-limited structure functions instead of the classical ones. On the other hand, Macek & Szczepaniak (2008) have developed a generalized two-scale weighted Cantor set using the partition function technique. Both methods are presented with emphasize on their advantages over the previous multifractal analyses. As an illustration, these new multifractal analyses are applied to a set of magnetic field data measured by Ulysses.

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INTRODUCTION

The solar wind is a highly nonlinear system and turbulent plasma characterized by strong intermittency. Fluctuating parameters such as velocity or magnetic field components alternate between bursts of activity and periods of quiescence. The classical way of emphasizing intermittency is by forming from the time series of the fluctuating quantity $X(t)$ a set of scale dependent time series $\delta X(t,\tau) = X(t+\tau) - X(t)$ for various time lag τ. We then construct the Probability Distribution Functions (PDFs) of the new time series $\delta X(t,\tau)$ for various scales τ. If the fluctuations are self-similar (or monofractal), the PDFs would collapse onto a single master curve P_s :

$$P(\delta X, \tau)\, \tau^s = P_s(\delta X/\tau^s) \tag{1}$$

where s is the scaling exponent [1]. On the other hand, if the fluctuations are intermittent (or multifractal), increasingly large extended tails are observed in the PDFs at smaller scales.

The first attempt to apply multifractal analysis to solar wind data was done in 1991 by Burlaga [2] using Voyager 2 data. Since then, the multifractal nature of the solar wind has been observed everywhere, from the inner to the outer heliosphere, at high heliographic latitudes and during various solar cycles.

Conventional multifractal analyses rely on the calculation of the structure functions which are moments of the PDFs. Because the structure functions incorporate the full set of fluctuation sizes, they are dominated by the fluctuations at the smallest sizes which are more numerous than large amplitude fluctuations. In addition, structure functions usually diverge for negative values of the moment order. Recently, Chang & Wu [3] have introduced the concept of rank-ordered multifractal analysis (ROMA). This procedure separates the multifractal characteristics of the minor fluctuations from the dominant background. The statistical properties of each isolated population can then be performed. The concept of range-limited structure functions (RLSF) introduced in this procedure solves the problem of divergence. Finally, the resulting multifractal spectrum obtained with this method can be explained in terms of the one-parameter scaling equation (1).

Another type of multifractal analysis is based on the computation of the partition function from which generalized dimensions D_q and the singularity spectrum can be extracted [4, 5]. The basic idea is to define an appropriate segmental measure $\mu_i(\tau)$ and calculate the partition function $\Gamma(q,\tau) = \sum \mu_i^q(\tau)$ for this measure.

We then search for the dominant singular behavior of $\Gamma(q,\tau)$ characterized by a power law with exponents $\gamma(q)$ for small τ. From the set of exponents $\gamma(q)$, the generalized dimensions $D_q = \gamma(q)/(q-1)$ can be obtained. The singularity spectrum $f(\alpha)$ can be obtained from $\gamma(q)$ using a Legendre transform.

Several theoretical models have been developed to reproduce the distributions of D_q and $f(\alpha)$. They have been applied to the solar wind to describe the nonuniform distribution of energy assuming a fully-developed turbu-

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lence and a cascade of energy. In particular, the p-model [6] cannot fit the observed spectrum of D_q for negative values of the index q and cannot produce an asymmetric singularity spectrum as observed in the solar wind. Macek & Szczepaniak (2008) [7] introduced a generalized two-scale weighted Cantor set model for solar wind as an extension of the classical p-model. Their model is able to fit the spectrum of D_q for negative q and predicts an asymmetric singularity spectrum [8].

RANK-ORDERED MULTIFRACTAL ANALYSIS (ROMA)

The main objective behind the ROMA is to individually carry out statistical analyses for subsets of the fluctuations with various fractal behaviors within the full multifractal set. The grouping of the fluctuations cannot be based only on the sizes of the fluctuations because the amplitudes of these fluctuations might be very different for different scales τ. Instead, we rank order the sizes of the fluctuations based on amplitudes of the scaled variable $Y = \delta X / \tau^s$. We choose ranges of ΔY in which the fluctuations of various time lags τ exhibit a monofractal behavior and are therefore characterized by a single exponent s. We repeat the procedure for each range ΔY covering the full set of the fluctuations and we obtain a multifractal spectrum $s(Y)$. The physical meaning of this spectrum is that it allows the full collapse of the PDFs on a single scaled PDF for all time lags considered.

To construct this rank-ordered multifractal spectrum, we sort the fluctuations into ranges of $\Delta Y = [Y_1, Y_2]$ and we evaluate the RLSFs of order q within each range :

$$S_q(|\delta X|, \tau) = \int_{a_1}^{a_2} |\delta X|^q P(|\delta X, \tau)\, d(|\delta X|) \qquad (2)$$

where $a_1 = Y_1 \tau^s$ and $a_2 = Y_2 \tau^s$. We search for a value of s such that those structure functions exhibit a monofractal behavior, i.e. $S_q \sim \tau^{sq}$ within each range of ΔY. Unlike the classical structure functions, the RLSFs also exist for negative values of q as long as the range ΔY does not include the value $Y = 0$.

TWO-SCALE CANTOR SET MODEL

In the two-scale weighted Cantor set model, an initial segment of length L is iteratively divided in two segments of unequal lengths l_1 and l_2 with the condition $l_1 + l_2 \leq L$. To both segments is associated a weight, respectively p_1 and p_2 with $p_1 + p_2 \leq 1$. After a sufficiently large number of iterations n, the resulting set is a strange attractor whose multifractal properties can be described by a set of generalized dimensions D_q and a singularity spectrum $f(\alpha)$.

In the solar wind, this model is used to describe the standard scenario of cascading eddies, l_1 and l_2 playing the role of typical sizes of the eddies. p_1 and p_2 are associated to the rate of the energy transfer between eddies. In the inertial range, there is no dissipation of energy, hence $p_1 + p_2 = 1$. However, the rate of energy transfer between eddies is inhomogenous and we have $p_1 = p$ and $p_2 = 1 - p$. Intermittency produced by this two-scale model is much larger than the one produced by the one-scale p-model, as evidenced by the more frequent occurence of large amplitude peaks [7].

DATA

To illustrate the two methods, we use a 21 days sample of the radial component of the magnetic field measured by Ulysses in 1994. The spacecraft was located at about 3.8 AU, at an heliographic latitude of $-50°$. Time resolution of the data varies between 1 and 2 seconds. Total number of points is 1.200.000. At this period and location, Ulysses was mainly located in fast wind streams during a solar minimum. The PDFs shown in Figure 1 clearly indicate that these solar wind magnetic field fluctuations are intermittent. At small scale ($\tau = 16\,$s), the PDF shows a clear departure from a Gaussian distribution with large extended tails while at large scales ($\tau = 2048\,$s), it is very close to a Gaussian distribution.

NEW MULTIFRACTAL ANALYSES

With the rank-ordered multifractal analysis

We start with a range of ΔY adapted to the size of the fluctuations $\delta B_R(\tau)$ for various scales τ. For example, we consider here $\Delta Y = [0.002, 0.004]$ for which the minimum and maximum values of the fluctuation sizes a_1 and a_2 can be calculated for each scale τ. We limit ourselves to scales τ for which the RLSFs vary linearly with τ (scales between $\tau = 4\,$s and $\tau = 1024\,$s). The RLSFs are then evaluated for 100 values of s regularly spaced between 0 and 1 and we search for the value(s) of s for which $\zeta(q) = sq$. An example of the RLSFs obtained with $s = 0.41$ is shown in Figure 2.

For each value of s, a set of scaling exponents $\zeta(q)$ is obtained. In principle, one can use a single value $q^\star$ to find out the value(s) of s for which $\zeta(q^\star) = sq^\star$. In practice, it is always better to apply this procedure to several values of q in order to minimize the influence of the statistics. In our example, we find a linear relation for $s_1 = 0.41$ and $s_2 = 0.74$. This is illustrated in Figure 3

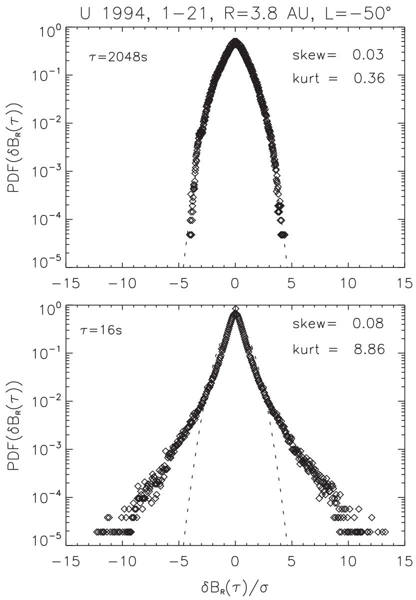

FIGURE 1. Probability distribution functions of fluctuations of the solar wind radial magnetic field for $\tau = 2048\,\mathrm{s}$ and for $\tau = 16\,\mathrm{s}$. The dashed line corresponds to a Gaussian distribution.

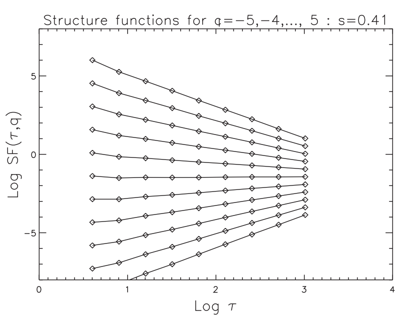

FIGURE 2. Log-log plot of the RLSFs as function of the scale τ. Structure fonctions are shown for values of q between -5 and 5 and for scales between $\tau = 4\,\mathrm{s}$ and $\tau = 1024\,\mathrm{s}$. The value of the scaling exponent s is 0.41. The slopes of the structure fonctions give $\zeta(q)$ for this particular choice of s.

for $q = 1$.

The same operations are repeated for many ranges ΔY of the scaled variable in order to cover the whole set of fluctuations $|\delta B_R(\tau)|$. For each range of ΔY, we obtain 2 multifractal spectra $s_1(Y)$ and $s_2(Y)$ which rescale the PDFs for the selected scales τ (see Figure 4). $s_1(Y)$ is approximately flat and in good agreement with the monofractal value $s = 0.42 \pm 0.02$ found by Hnat et al [1] for the B^2 fluctuations based on WIND data. The bifurcated solution $s_2(Y)$ for values of $Y \leq 0.01$ might be the result of some bending of the RLSFs at large scales of τ indicating the possible mixing of fluctuations belonging to two regimes of scales with different multifrac-

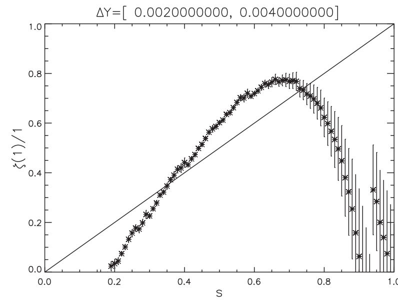

FIGURE 3. Plot of $\zeta(q)/q$ as a function of s for $q = 1$ and 100 values of s regularly spaced between 0 and 1. The straight line corresponds to a linear variation in s with slope 1. Error bars are plotted. The large spread of the points on the right side is due to low statistics.

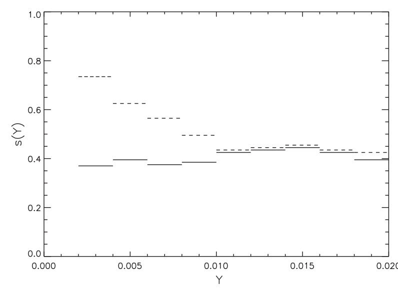

FIGURE 4. Multifractal spectra $s(Y)$ obtained with the Ulysses data for $Y = [0.002, 0.02]$ with $\Delta Y = 0.002.s_1$ is the solid line, s_2 is the dashed line.

tal characteristics, a behavior similar to that considered recently by Tam et al (2009) [9] for ionospheric electric field fluctuations. Another possibility might be poor statistics which would result in an inconclusive evidence of PDF collapse for the second bifurcated spectrum. We plan to study these possibilities for a future publication.

With the two-scale Cantor set model

First, we construct a multifractal measure associated with magnetic field fluctuations on scale τ, $\mu_i(t) = \varepsilon_i(t)/\sum_{i=1}^{M}\varepsilon_i(t)$ where $\varepsilon(t) = |B_R(t+\tau) - B_R(t)|^2$ and M is the total segments of length τ in the total time interval. Below, we use spatial scales l instead of τ because Taylor's hypothesis applies in the solar wind.

Second, for a given value of the moment order q, we calculate the partition function :

$$\Gamma(q,l) = \sum \mu_i^q \sim l^{D_q(q-1)} \tag{3}$$

The slopes of the logarithm of $\overline{\mu}(q,l) = \sqrt[q-1]{\Gamma(q,l)}$ versus $\log l$ provides the generalized dimensions $D_q(l)$.

This is illustrated in Figure 5 for time lags between 4 s and 2048 s. The corresponding values of D_q are shown

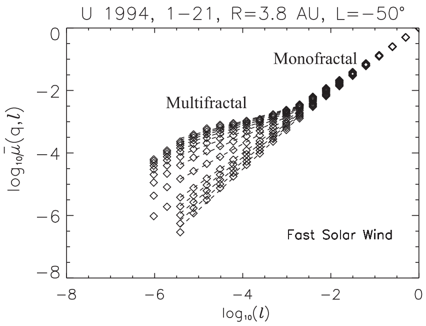

FIGURE 5. Log-log plot of $\overline{\mu}(q,l)$ as a function of the scale l. The order moment q shown varies between -2 and 8. Time lags are in between $\tau = 4\,\mathrm{s}$ and $\tau = 2048\,\mathrm{s}$.

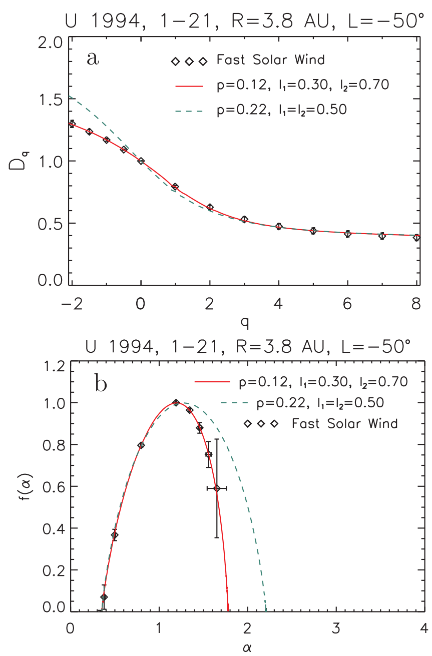

FIGURE 6. (a) The generalized dimensions D_q as a function of the moment order q and (b) the singularity spectrum $f(\alpha)$ for the magnetic field fluctuations observed by Ulysses. Diamonds correspond to values obtained using Ulysses data. The dotted line is the best fit obtained with the one-scale p-model while the solid line is the best fit obtained with the two-scale model.

in Figure 6a as well as the fits with the one-scale p-model and the two-scale model. The two-scale model with parameters $p = 0.12, l_1 = 0.30$ and $l_2 = 0.70$ is in very good agreement with the spectrum of D_q obtained from the data.

The singularity spectrum is obtained using the Legendre transform and is shown in Figure 6b. Again, data are better fitted with the two-scale model. In particular, the model can reproduce the asymmetric form of the spectrum. A measure of this asymmetry is given by $A = \frac{\alpha_0 - \alpha_{\min}}{\alpha_{\max} - \alpha_0}$ where $f(\alpha_0) = 1$ and $\alpha_{\min}$ and $\alpha_{\max}$ are the minimum and maximum values of the spectrum. For Ulysses data, we find $A = 1.33$.

CONCLUSIONS

To summarize, we have presented two recent multifractal models which better characterize the multifractal nature of intermittent fluctuations. The rank-ordered multifractal analysis solves the problems inherent to the classical use of structure functions and provides a physical interpretation of the resulting multifractal spectrum. The generalized two-scale weighted Cantor set model generalizes the classical p-model and provides better fits of the spectrum of generalized dimensions and of the singularity spectrum. Moreover, it gives us information about degree of multifractality and asymmetry.

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