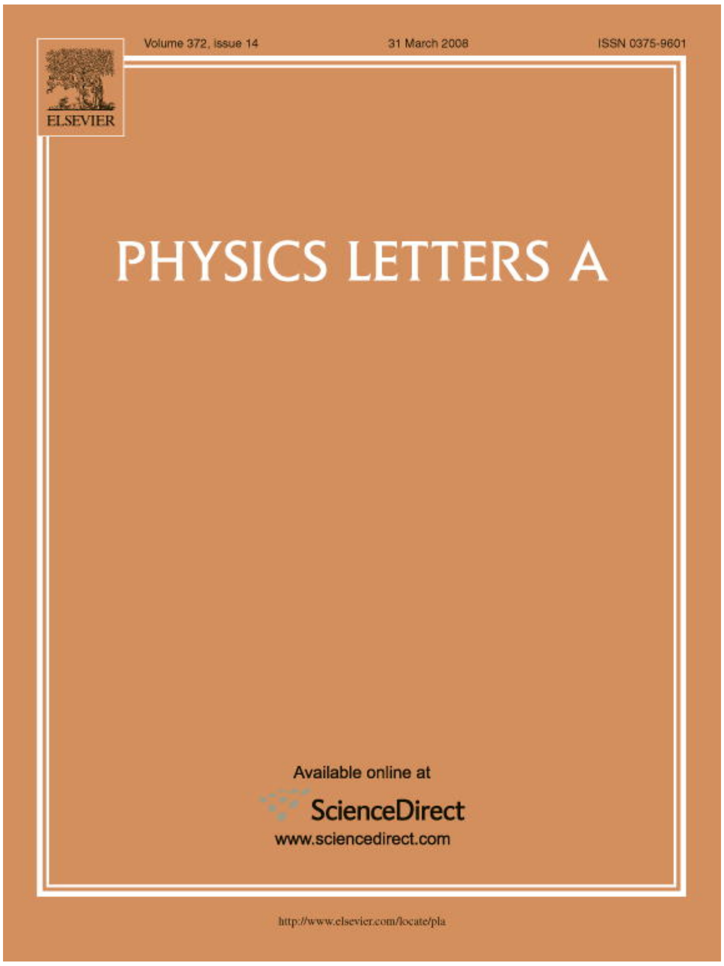
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Unstable manifolds for the hyperchaotic Rössler system

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Abstract

We analyse stability of the generalized four-variable Rössler oscillating system depending on selected control parameters, by using analytic and Hurwitz–Routh methods. In contrast to the usual three-dimensional Rössler and Lorenz systems, we show that always there exists at least one unstable direction, and the number of positive local Lyapunov exponents may be different for both fixed points. We have found two new types of Hopf bifurcation, in which the dimension of the unstable manifold can be increased or reduced by two. Hence there are many possibilities for hyperchaotic unstable manifolds of various dimensions. We have also calculated various ranges of the control parameters for which different unstable manifolds can be obtained. This allows a better characterization of stability of the attractors in the hyperchaotic regime.

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Keywords: Nonlinear dynamics and nonlinear dynamical systems; Chaotic dynamics

1. Introduction

Chaos is a typical behaviour of nonlinear dynamical systems that exhibits sensitivity to initial conditions. It is worth noting that usually the Lyapunov exponents are provided by the real parts of the eigenvalues of the Jacobian matrix of the system. Hence the main issue in such a system is stability. If one eigenvalue of its characteristic equation is positive, then the system has locally one unstable direction and the corresponding fundamental solution would increase exponentially. This instability could be balanced by shrinking in another stable direction, besides possible marginal stability with one zero eigenvalue. The usual Lorenz [1] and Rössler systems [2] are well known examples of this kind of dynamical systems. Hence for continuous systems at least three dimensions are necessary for chaotic behaviour with stretching and folding properties of trajectories. Therefore, a hyperchaotic behaviour with two positive Lyapunov exponents is only possible for at least four-dimensional system. An example of such a system oscillating hyperchaotically has also been presented by Rössler [3]. However, the stability of the hyperchaotic system is more complex and a comprehensive analysis of its dependence on the systems parameters is still missing in the literature.

In order to analyse stability of the system we have to find steady state (equilibrium) points and look for the eigenvalues of the Jacobian matrix analysing the resulting characteristic equations for each equilibrium point. The solutions of three-dimensional polynomials can easily be obtained analytically using Cardano formulae, but for higher-order systems the proper methods are provided by Hurwitz [4] and Routh [5] criteria. Admittedly, in case of a four-dimensional system we can still use analytical formulae, but only for some limited ranges of parameters. Therefore, in this Letter we use both analytical and numerical methods to analyse stability of the Rössler oscillator and examine various ranges of the control parameters for which different dimensions of unstable manifolds can be obtained.

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2. Hyperchaotic system

Rössler presented an oscillating system containing only one nonlinear term, but producing chaos with two directions of hyperbolic instability on the attractor. The generalized systems have also been studied (e.g., [6–8]). In particular, Baier and Sahle [6] proposed the generalized Rössler system (GRS), obtained by linearly coupling additional degrees of freedom to the original three-dimensional Rössler system [2]. The system obtained in this way exhibits chaotic and hyperchaotic behaviour. Meyer et al. [7] introduced a mode transformation of the GRS based on the solutions of the linear subsystem, which have been used to analyse dynamics of the GRS. In order to understand hyperchaotic dynamics Nikolov and Clodong [8] considered some modified hyperchaotic Rössler systems (MHRS), which were obtained after introducing parameter b in the z equations $(b \to b + b_1 x(t) + b_2 y(t) + b_3 z(t) + b_4 w(t))$ [9]. They have considered how the change of the type of the fixed points influences the prediction time in MHRS. Therefore, the modifications and generalization of the hyperchaotic Rössler model are well described in literature, while the dependence of this system on control parameters is still poorly known. The aim of our work is to provide the information about stability of the system near equilibrium points as a function of control parameters. Here we deal with the following equations:

$$\begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + ay + w, \\ \dot{z} = b + xz, \\ \dot{w} = -cz + dw. \end{cases} \tag{1}$$

This system can simulate a chemical reaction scheme for the following values of parameters: $a = 0.25$, $b = 3.0$, $c = 0.5$, and $d = 0.05$, as described by Rössler [3]. Numerical methods show that the attractor has two positive Lyapunov exponents, $\lambda_1 = 0.11$, $\lambda_2 = 0.02$, Ref. [10]. An efficient feedback control has also be designed for this system [11].

3. Hurwitz–Routh and analytic methods

We use Hurwitz and Routh methods in order to analyse stability of the four-variable Rössler system depending on control parameters. First, Hurwitz method informs us when all the roots of the characteristic polynomial $P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda^1 + a_0$ with real coefficients have negative real parts [4]. In this theorem we have to check when two conditions are satisfied. A necessary but not sufficient condition is that all coefficients of polynomial (a_i for $i = 0, \ldots, n$) have the same sign. A necessary and sufficient condition is that all the principal leading minors of the Hurwitz matrix are strictly positive. In the Hurwitz matrix the coefficients of the characteristic polynomial are given on the main diagonal. All the other entries of the matrix corresponding to coefficients with subscripts greater than degree of polynomial or less than zero are set equal to zero. If these two conditions are fulfilled, then the characteristic polynomial is called Hurwitz, i.e., it has all its roots on the left-hand complex plane. When conditions of Hurwitz criterion are not satisfied, we get information that some eigenvalues have positive real parts and hence the system is unstable. Second, another Routh method [5] gives us an accurate information about the number of positive eigenvalues. In this method the number of roots on the right-hand plane (real part greater than zero) is equal to the number of sign changes in the first column of the following Routh array:

$$\begin{array}{l|lllll} \lambda^n & a_n & a_{n-2} & a_{n-4} & \ldots & a_0 \\ \lambda^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots & 0 \\ \lambda^{n-2} & b_{n-2} & b_{n-4} & \ldots & & \\ \lambda^{n-3} & c_{n-1} & c_{n-3} & \ldots & & \\ . & . & & & & \\ . & . & & & & \\ . & . & & & & \\ \lambda^0 & . & & & & \end{array}$$

where

$$b_{n-i} = \frac{a_{n-1}a_{n-i} - a_n a_{n-i-1}}{a_{n-1}},$$
$$c_{n-i} = \frac{b_{n-2}a_{n-i-2} - a_{n-1}b_{n-i-3}}{b_{n-2}}.$$

The Routh array for n-dimensional polynomial has $n+1$ rows. Each new row is obtained from the two rows immediately above it. Zeros in the first column represent roots on the imaginary axis. Therefore, we use the combined Hurwitz–Routh stability criterion as a method for determining as to whether or not a given system is stable and what are the dimensions of unstable manifolds. The Hurwitz–Routh method based on coefficients in the characteristic equation of the system is particularly useful for high-order systems, because it does not require to find eigenvalues and still gives us important information about stability, e.g., [6]. Fortunately, in the case of four-variable Rössler system we are still able to solve analytically characteristic equations at least for some range of parameters. Analytic considerations consist of conversion of the normalized fourth degree polynomial $P(\lambda)$ into two quadratic polynomials

$$P_{1,2} = \lambda^2 + \left(a_3 \pm \sqrt{8y + a_3^2 - 4a_2}\right)\frac{\lambda}{2} + \left(y \pm \frac{a_3 y - a_1}{\sqrt{8y + a_3^2 - 4a_2}}\right)$$

and finding the corresponding roots. Here y is the solution of the following equation:

$$8y^3 - 4a_2y^2 + (2a_3a_1 - 8a_0)y + a_0\left(4a_2 - a_3^2\right) - a_1^2 = 0. \tag{2}$$

We use analytic method only when Eq. (2) has real solutions for y. Finally, we compare eigenvalues obtained analytically with the numerical results and test applicability of the Hurwitz–Routh method.

4. Bifurcations and unstable manifolds

For each of the two equilibrium points, where the first point is denoted by S_+ and second by S_-, we have the characteristic polynomial, providing that $b \geqslant 0$, $d > 0$ and $c > ad$. In particular, for the standard values: $a = 0.25$, $b = 3.0$, $c = 0.5$, and

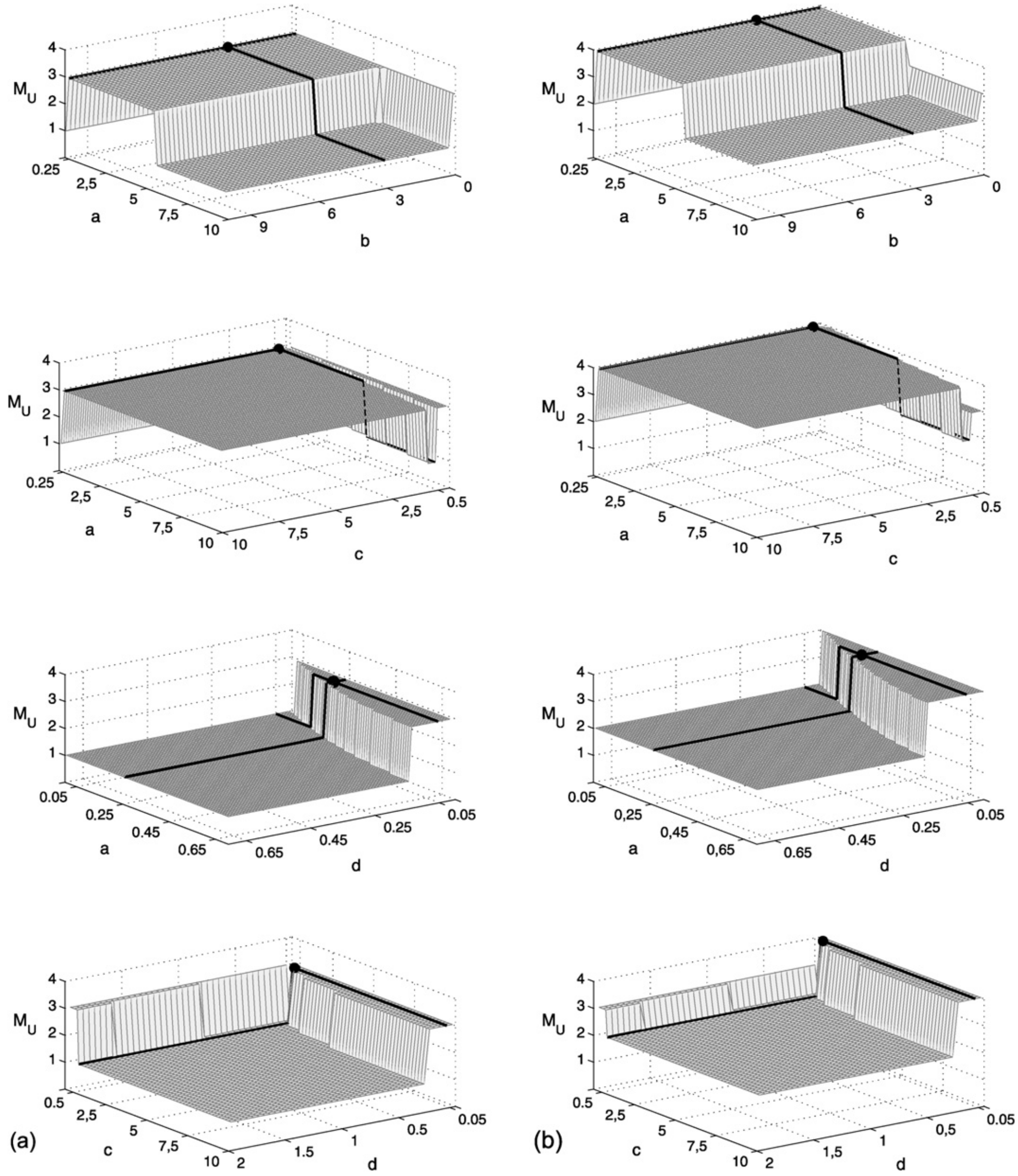

Fig. 1. The dimension M_U of unstable manifolds for (a) S_+ and (b) S_-.

$d = 0.05$, the obtained eigenvalues for the first and second characteristic polynomial are: $\lambda_1 = 0.0493731 - 0.998687i$, $\lambda_2 = 0.0493731 + 0.998687i$, $\lambda_3 = 0.101891$, $\lambda_4 = -5.30896$ and $\lambda_1 = 0.0501792 - 0.971052i$, $\lambda_2 = 0.0501792 + 0.971052i$, $\lambda_3 = 0.103929$, $\lambda_4 = 5.50404$, respectively.

In Fig. 1(a) and (b) we show the dimension M_U of unstable manifolds as a function of two selected control parameters for S_+ or S_-, respectively; the standard values are taken for the other two fixed parameters. The dimension of unstable manifolds for standard values of control parameters has been denoted by black points. The black lines show how M_U depends on the change of control parameters. The values of parameters for Hopf bifurcation points (subscript H) with purely imaginary eigenvalues ($\mathrm{Re}\,\lambda = 0$), which characterize behaviour of spirals and limit cycles that appear at bifurcation, are given in Table 1.

Table 1
Values of the parameters for Hopf bifurcation ($\mathrm{Re}\,\lambda = 0$)

	S_+	S_-	ΔM_U
a_{H1}	0.14859	0.14708	2
a_{H2}	5.58503	5.52458	−2
c_H	0.25856	0.23845	2
d_H	0.08268	0.08428	−2

Let us start with parameter $b \geqslant 0$ that is present together with the nonlinear term in Eq. (1); this parameter controls the depth of spiraling of the attractor along one axis, as has been demonstrated in the original Rössler paper [3, Fig. 1]. The dependence of the dimension of unstable manifold on b is rather simple. For example, for $a = 0.25$ we have always three positive eigenvalues, $M_U = 3$ for S_+, and four positive eigenvalues,

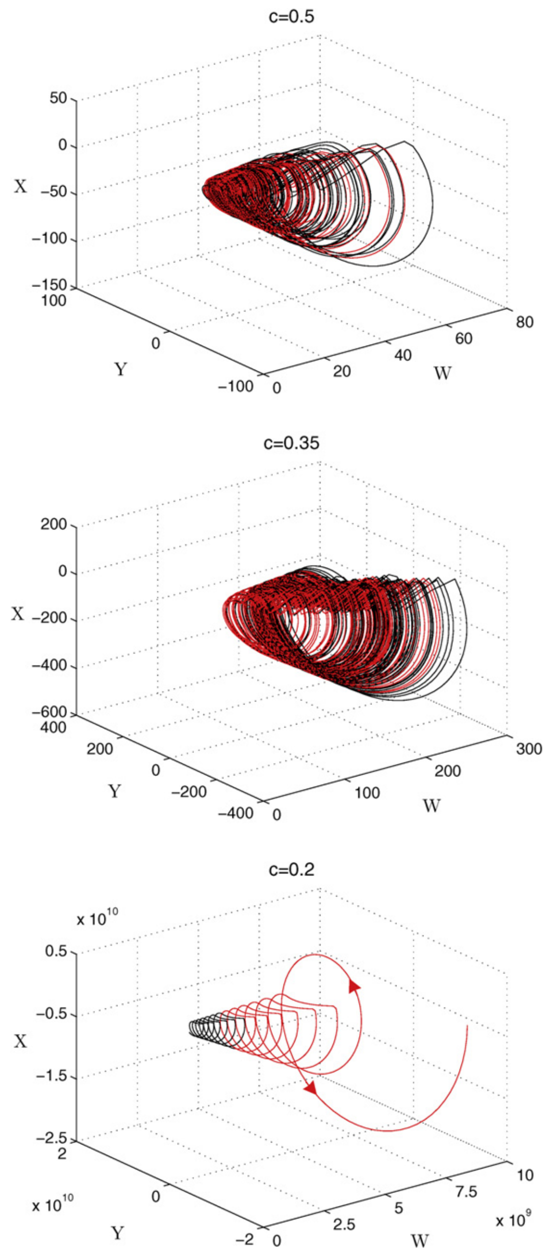

Fig. 2. Phase portraits of the hyperchaotic Rössler system in XYW subspace.

$M_U = 4$ for S_-. Naturally, for different value of a this result can be modified, but there is no change of the character of stability (no bifurcation points) in the whole range of b, if the other parameters are fixed. In addition, the analytical method could be applied in the full range of this parameter. In particular, for $b = 0$ we have only one equilibrium point $x = y = z = w = 0$ with one zero eigenvalue, one real and positive, and the other complex eigenvalues with positive real parts.

On the contrary, for the next parameter a, where $0 < a < c/d = 10$, the analytic method is only restricted to some narrow part of this full range, namely $(0, 1.827) \cup (8.415, 10)$, where there exist real roots of Eq. (2), and here the situation is much more complicated. Nevertheless, using Hurwitz–Routh method, we have obtained the following two values of Hopf bifurcation points: $a^+_{H1} = 0.149$, $a^+_{H2} = 5.585$ for S_+, or $a^-_{H1} = 0.147$, $a^-_{H2} = 5.525$ for S_-, respectively, as given in Table 1. Therefore, as seen in Fig. 1, we have one (or two) positive eigenvalues for $a \in (0, a^{\pm}_{H1})$, three (or four) positive eigenvalues for $a \in (a^{\pm}_{H1}, a^{\pm}_{H2})$, and again one (or two) for $a \in (a^{\pm}_{H2}, 10)$ for S_+ (or S_-).

On the other hand, for parameter d the analytic method is again available for full range of parameters, where $d < c/a = 2$, and we have only one bifurcation point. Here the unstable manifold of dimension three, $M_U = 3$ (or four, $M_U = 4$), is reduced into one- (or two-)dimensional manifold, $M_U = 1$ (or $M_U = 2$) for S_+ (or S_-), i.e., the dimension of unstable manifold is reduced by two at the bifurcation point ($\Delta M_U = -2$). The situation is similar for $c > ad = 0.0125$, except that the dimension of the unstable manifold is now increased by two at the corresponding bifurcation point ($\Delta M_U = 2$). In summary, we see that our combined method is able to give full information about stability of hyperchaotic dynamical systems near equilibrium points.

It is worth noting that now there is no room for stability, in contrast to the usual three-dimensional Lorenz [1] and Rössler [2] systems, where the number of positive local Lyapunov exponents is the same for both fixed points. However, here for our four-dimensional system in the whole range of parameters (for $a > 0$ for outgoing spirals) there is always at least one unstable direction ($M_U \geqslant 1$, Ref. [12]), providing that the equilibrium points exist; one could only have $M_U = 1$ or 3 for S_+, and $M_U = 2$ or 4 for S_-. Now, for each fixed point the number of positive local Lyapunov exponents is determined by the dimension of the corresponding unstable manifold. We have one or two positive exponents for S_+ and two or three for S_-. Only for a small range of parameter a from $a^-_{H2} = 5.52458$ to $a^+_{H2} = 5.58503$ we have two positive exponents for both fixed points.

Moreover, two new types of Hopf bifurcation appear, as listed in Table 1: one in which the dimension of unstable manifold is increased, $\Delta M_U = 2$, and the other where this dimension is reduced, $\Delta M_U = -2$. Hence there are many possibilities for unstable higher-dimensional hyperchaotic manifolds, as demonstrated in Fig. 1.

5. Hyperchaotic attractor

Now, we analyse the structure of the hyperchaotic attractor for the values of the control parameter c at which different dimensions of unstable manifolds exists (below and above values for Hopf bifurcation). The method used by us is a quasi-constant step size implementation of the numerical differentiation formulas (NDF) expressed in terms of backward differences (also known as Gear's method) [13]. Contrary to the well-known Runge–Kutta technique, these formulas are the implicit methods, which allows for an effective reduction of time step, when the solution varies rapidly. The obtained results are presented in Fig. 2. The first upper part of this figure shows the attractor's trajectories for the standard values of control parameters [3], while below we present the other phase portraits for two other values of parameter c, namely: $c = 0.35$ and $c = 0.2$, i.e., above and below the characteristic values for Hopf bifurcations, correspondingly (cf. Table 1). For illustration of the stretching and folding properties of trajectories on the attractor, in two upper views we have depicted trajectories that are in black for the time interval of 4500–5000 s, and for a latter time period of 17500–18000 s they are marked in red. This clearly shows that the same regions of the attractor are visited at various times by trajectories of this system. On the other hand, in the lower image for $c = 0.2$, i.e., below Hopf bifurcation, we have a trajectory for a time interval of 550–600 s (black line) followed by a red line during a later period of 600–650 s (in this case it is enough to integrate the system equations for much smaller time interval of 750 s). As we see in this latter case we have a simple outgoing spiral in phase space WYX in the direction shown by arrows. On the contrary, for the value of parameter c above the Hopf bifurcation a "contracting" structure of the attractor actually exists for a certain range of the control parameter, e.g., for $c = 0.5$ and $c = 0.35$. We can expect that difference of behaviour in the former and latter case is a result of new types of bifurcations, characterized by a change of the dimension of unstable manifolds at $c = 0.23845$ (in case of the fixed point S_-) and $c = 0.25856$ for S_+, as given in Table 1.

6. Conclusions

The question of stability of the four-dimensional Rössler system near equilibrium points can be fully resolved by using the combined Hurwitz–Routh and analytical methods. In contrast to the usual three-dimensional Rössler and Lorenz systems, we have shown that there always exists at least one unstable direction and there are many possibilities for more than one-dimensional hyperchaotic unstable manifolds, and the number of positive local Lyapunov exponents may be different for both fixed points. It is worth noting that two new types of transitions appear at Hopf bifurcations, in which the dimension of the unstable manifold is increased or decreased by two, corresponding to the change of the number of positive local Lyapunov exponents by one. We expect that for higher order dynamical systems the situation could be even more complicated. But we still recommend Hurwitz and Routh methods because they do not require of analytical calculation of the roots of the characteristic polynomials. We also hope that the Rössler oscillator could be a useful model in many real systems, for example, for the complex solar wind plasma as discussed, e.g., in [14,15].

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