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Influence of colored noise on chaotic systems

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We focus on classical chaotic systems corrupted by white and colored noise. We study the dependence of the correlation dimension and the Kolmogorov entropy on the noise level and its spectral exponent. As is well known, white noise strongly reduces the width of the scaling region for the correlation dimension and entropy. On the contrary, we demonstrate that colored noise does not basically obscure the scaling region, changing only the shape of the correlation sum for length scales smaller than the noise level. The numerical results show that, even for a noise level as high as $\sim 5\%$, a reasonably wide plateau for the correlation sum is still obtained, but the value of the calculated dimension is somewhat increased. The calculated correlation dimension is a bilinear function of the noise level and the dimension of the noise, which depends on the spectral exponent of the noise. On the other hand, the width of the scaling region for the correlation entropy depends on this spectral exponent, but the value of the plateau does not change substantially.

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In order to detect and quantify chaos in a dynamical system, it is necessary to deal with a cleaned signal. Both the correlation dimension and the Kolmogorov entropy are sensitive to the presence of small amounts of noise, which may obscure the underlying fractal structure [1–6]. White noise basically tends to fill all the phase space available, providing an infinite correlation dimension. On the contrary, colored noise has a finite correlation dimension, which depends on the spectral exponent α [7,8]. Thus, we expect that the influence of colored noise on low-dimensional chaotic systems will be different from that of white noise. Here we analyze the following three classical models contaminated by white and colored noise: (1) a discrete chaotic Hénon map ($a=1.4$, $b=0.3$) [9], and two continuous chaotic systems generated (2) by Rössler equations ($a=0.15$, $b=0.2$, $c=10$) [10], with sampling time $\Delta t=0.15$ s and delay time $\tau=10\Delta t$, and (3) by Lorenz equations ($\Sigma=16.0$, $R=45.92$, $b=4.0$) [10], with sampling time $\Delta t=0.15$ s and delay time $\tau=4\Delta t$. The selected parameter for the three models were chosen to represent chaotic dynamics. We take time series of x components consisting of $N=16\,384$ points.

In this study we add white and colored noise to these chaotic systems, in order to study the dependence of the correlation dimension and the Kolmogorov entropy on the noise level σ and the spectral exponent α. Namely, we like to extend the results known in the case of white noise contamination of chaotic data [3,11] to the case of colored noise.

In order to compute the correlation dimension and Kolmogorov entropy we make use of the Grassberger and Procaccia method [9]. Therefore, we first briefly review this procedure. Using time series of equally spaced data, we construct a large number of vectors $\mathbf{X}(t_i)=(x(t_i),x(t_i+\tau),\ldots,x(t_i+(m-1)\tau))$ in the embedding phase space of dimension m, where $i=1,\ldots,n$ with $n=N-(m-1)\tau$. Then, we divide this space into a large number $M(r)$ of equal hypercubes of size r, which cover the presumed attractor. If p_j is the probability measure that a point from a time series falls in a typical jth hypercube, using the q-order function $I_q(r)=\Sigma(p_j)^q, j=1,\ldots,M$, the generalized dimension [12–14] is, e.g.,

$$D_q=\frac{1}{q-1}\lim_{r\to 0}\,[\ln I_q(r)/\ln r]. \tag{1}$$

The related q-order Rényi-Kolmogorov information entropy [9,13,14] is given by

$$K_q=\lim_{r\to 0}\lim_{m\to\infty}\frac{1}{1-q}\ln I_q(r). \tag{2}$$

In practice, $q=2$ is sufficient and $I_2(r)$ is taken to be equal to the correlation sum [15]

$$C_m(r)=\frac{1}{n_{\mathrm{ref}}}\sum_{i=1}^{n_{\mathrm{ref}}}\frac{1}{n-2n_{\mathrm{c}}-1}\sum_{j=n_c+1}^{n}\theta(r-|\mathbf{X}(t_i)-\mathbf{X}(t_j)|), \tag{3}$$

with $\theta(x)$ being the unit step function, where $n_{\mathrm{ref}}=500$ is the number of reference vectors and $n_c=4$ is Theiler's correction [16]. For large m and small r in the scaling region it can be argued that

$$C_m(r)\propto r^{D_2}e^{-m\,K_2}, \tag{4}$$

where D_2 and K_2 are approximations of the ideal $r\to 0$ and $m\to\infty$ limits in Eq. (2) for $q=2$, Refs. [9,15].

The next point is to generate colored noise with various spectral exponents α. Following the Osborne and Provenzale method [7,8,17], we consider a stochastic process described by the standard Fourier series:

$$X(t_i)=\sum_{k=1}^{M/2}\zeta_k\cos(\omega_k t_i+\phi_k), \tag{5}$$

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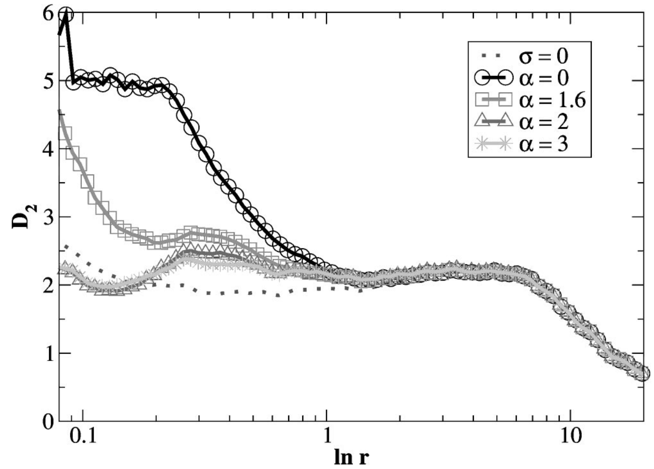

FIG. 1. The function $D_{2,\mathrm{m}}(r)=d[\ln C_m(r)]/d(\ln r)$ versus $\ln r$ obtained for the Rössler attractor with noise level $\sigma=2\%$ is shown for fixed embedding dimension $m=8$ and various spectral exponents α.

where ϕ_k are the random phases, computed at times $t_i=i\Delta t$, with $i=1,\ldots,M$ $(T=M\Delta t)$, and frequencies equal to $\omega_k=k\Delta\omega$, with $k=1,\ldots,M/2$ $(\Delta\omega=2\pi/T)$. The coefficients ζ_k are related to the power spectrum $P(\omega_k)$ of the random function by

$$\zeta_k=[P(\omega_k)\Delta\omega]^{1/2}. \tag{6}$$

If the power spectrum has a power-law dependence

$$P(\omega_k)=C\omega_k^{-\alpha}, \tag{7}$$

with random phases ϕ_k uniformly distributed on the interval $[0,2\pi)$ for a given spectral exponent $1<\alpha\leqslant 3$, then the resulting signal is self-affine. It follows that, after averaging (for a fixed value of α), we have

$$\langle|X(t_i+\Delta t)-X(t_i)|\rangle=\lambda^D\langle|X(t_i+\lambda\Delta t)-X(t_i)|\rangle. \tag{8}$$

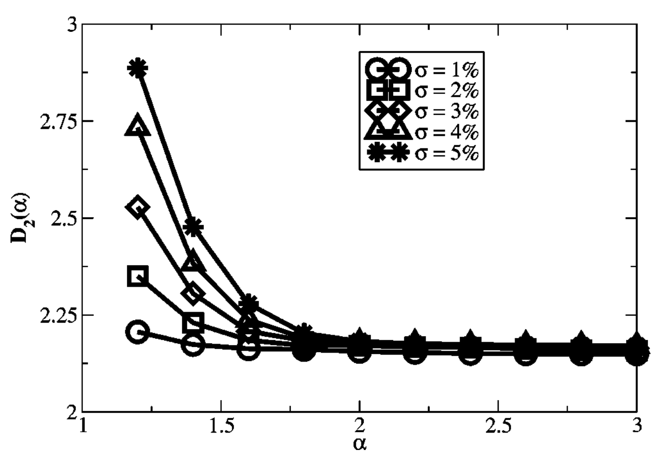

FIG. 2. The calculated correlation dimension $D_2(\alpha)$ versus α obtained for the Rössler attractor is shown for noise level $1\%\leqslant\sigma\leqslant 5\%$.

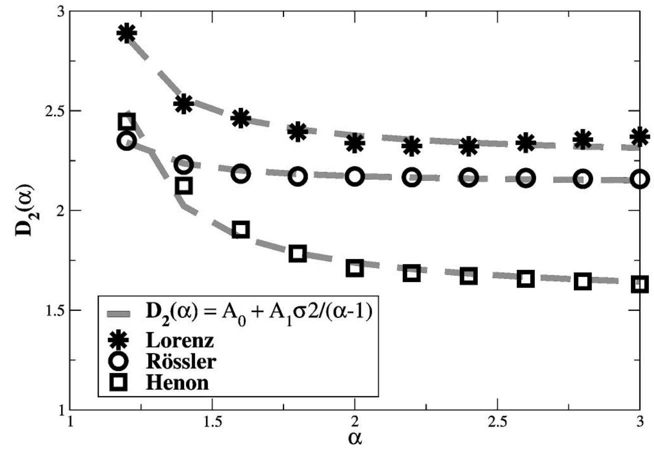

FIG. 3. The calculated correlation dimension $D_2(\alpha)$ versus α obtained for the Rössler, Lorenz, and Hénon attractors is shown for noise level $\sigma=2\%$. The dashed lines show the fitted function $D_2(\alpha)=A_0+A_1\sigma\,2/(\alpha-1)$.

The correlation dimension of the random signal, $D=D_2^{\mathrm{noise}}$, is then related to the spectral exponent α by

$$D_2^{\mathrm{noise}}(\alpha)=\frac{2}{\alpha-1}. \tag{9}$$

Now, we discuss the influence of noise in detail as follows. First, we calculate the correlation dimension $D_{2,\mathrm{m}}(r)=d[\ln C_m(r)]/d(\ln r)$ versus $\ln r$ for the Rössler, Lorenz, and Hénon attractors, corrupted by 2% of colored noise with various spectral exponents: $\alpha=0,1.6,2,$ and 3. In Fig. 1 we show the results of our calculations for the Rössler attractor. As discussed in Ref. [11], usually one can distinguish different types of behavior of $D_{2,\mathrm{m}}(r)$ for different regions of scales r. Small values of r are usually dominated by the noise

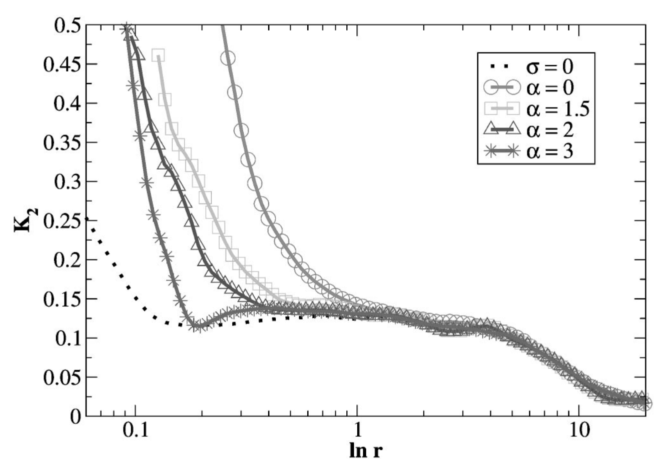

FIG. 4. The function $K_{2,\mathrm{m}}(r)=(1/\Delta m)\ln[C_m(r)/C_{m+\Delta m}(r)]$ in units of time lag $T=\tau\Delta_t$ versus $\ln r$ $(\Delta m=1)$ for the Rössler attractor with noise level $\sigma=2\%$ is shown for fixed embedding dimension $m=13$ and various spectral exponents α.

TABLE I. The parameters A_0 and A_1 of the fitted function $D_2 = A_0 + \sigma A_1 2/(\alpha - 1)$, for noise level $0.01 \leqslant \sigma \leqslant 0.05$ and spectral exponent $1 < \alpha \leqslant 3$.

System	A_0	A_1
Rössler	2.08 ± 0.06	1.9 ± 0.1
Lorenz	2.24 ± 0.15	3.1 ± 0.2
Hénon	1.5 ± 0.25	2.9 ± 0.4

in the data and, in the case of white noise, we expect that $D_{2,\mathrm{m}}(r)$ should be proportional to the embedding dimension m. Further, in the proper scaling region of distances r, if the D-dimensional attractor exists, we expect a plateau of the function $D_{2,\mathrm{m}}(r)$ for $m \geqslant D$ [18] and in the worst case for $m > 2D$ [19]; the plateau independent of m should provide the proper correlated dimension. In our case, embedding dimension is fixed to $m = 8$ (which is even larger than $m > 2D$), in order to demonstrate more clearly the influence of noise on the estimated dimension.

As seen in Fig. 1, for the case without noise ($\sigma = 0$) we obtain a wide and clear plateau for a scaling region of $0.1 < \ln r < 10$. The average of the function $D_2(r)$ in this range of r yields the proper correlation dimension for the Rössler attractor, $D_2 = 2.01 \pm 0.06$, Ref. [10]. In the presence of white noise ($\alpha = 0$) the width of the scaling region is strongly reduced, i.e., by one order of magnitude. On the contrary, colored noise ($1 < \alpha \leqslant 3$) does not basically obscure the scaling region for the dimension, changing only the shape of the correlation sum for length scales smaller than the noise level. A reasonably wide plateau for the correlation sum is still obtained, but the calculated dimension is somewhat increased. We also observe that the value of the plateau depend on the spectral exponent of colored noise α. In order to quantify the increase of the correlation dimension, we fix the scaling region $0.1 < \ln r < 10$ (as in the case of uncontaminated data), and then calculate the correlation dimension versus α, for noise level $1\% \leqslant \sigma \leqslant 5\%$ and spectral exponent $1 < \alpha \leqslant 3$.

The results of these calculations for the Rössler attractor are shown in Fig. 2. For increasing noise level σ, the calculated correlation dimension $D_2(\alpha)$ shows very similar behavior to the correlation dimension of the pure colored noise $D_2^{\mathrm{noise}}(\alpha) = 2/(\alpha - 1)$. Therefore, it makes sense to fit these points with a function that depends on the dimension of noise D_2^{noise} and the noise level σ. We try the simple bilinear function $D_2(\alpha) = A_0 + A_1 \sigma\, 2/(\alpha - 1)$ with two parameters, A_0 and A_1.

The fitted functions $D_2(\alpha) = A_0 + A_1 \sigma\, 2/(\alpha - 1)$ obtained for the Rössler, Lorenz, and Hénon attractors are shown in Fig. 3, for noise level $\sigma = 2\%$. Table I summarizes calculated values of parameters A_0 and A_1, for noise level $1\% \leqslant \sigma \leqslant 5\%$ and spectral exponent $1 < \alpha \leqslant 3$. The parameter A_0 should give the proper correlation dimension for the system not corrupted by noise. Admittedly, the values of these parameters are somewhat larger. Nevertheless, this bilinear function of the dimension of noise D_2^{noise} and the noise level σ fits quite well the values of the correlation dimension D_2 obtained for noise level as high as $\sigma \sim 5\%$. For larger amounts of noise, some deviations from the fitted function are observed. We suppose that, in this case, it is no more adequate to consider colored noise as a linear addition to the deterministic chaotic system. We should rather try to describe the system as a mixture of two components (a random fractal and a chaotic deterministic part).

For a larger number of points, $N = 32\,768$, we obtain essentially the same results. Again, the calculated correlation dimension $D_2(\alpha)$ is well approximated by the same bilinear function. In addition, the values of the parameter A_0 agree even better with the proper correlation dimensions for the systems not corrupted by noise. Namely, we obtain for the Rössler, Lorenz, and Hénon attractors, $A_0 = 1.97 \pm 0.06$, 1.96 ± 0.07, and 1.44 ± 0.14, correspondingly. The values of the parameter A_1 change within the errors given in Table I.

These numerical results also confirm that, when we deal with real data, the sole estimation of a finite correlation dimension, from the analysis of an unknown *a priori* system, is not sufficient to infer the presence of deterministic chaos in the system. On the other hand, the estimated correlation entropy does not depend on the spectral exponent of the colored noise α. In Fig. 4 the function $K_{2,\mathrm{m}}(r) = (1/\Delta m)\ln[C_m(r)/C_{m+\Delta m}(r)]$ versus $\ln r$ ($\Delta m = 1$) for the Rössler attractor with noise level $\sigma = 2\%$ is shown for various spectral exponents α. We see that for larger α a wider scaling region is obtained, but the value of the plateau does not basically change, providing a finite and positive correlation entropy. Since we know that the correlation entropy of colored noise converges to zero [17], we suggest that the estimation of the correlation entropy is a more suitable method able to distinguish between chaos and colored random noise. For the correlation entropy, the dependence of the width of the scaling region on the spectral exponent of the noise α will be extensively discussed in the forthcoming paper.

To conclude, white noise strongly reduces the width of the scaling region for the correlation dimension and entropy. On the contrary, colored noise does not basically obscure the scaling region for the dimension, changing only the shape of the correlation sum for length scales smaller than the noise level. For some amounts of colored noise (even for a noise level as high as $\sigma \sim 5\%$), a reasonably wide plateau for the correlation sum is still obtained, but the calculated dimension is somewhat increased. The obtained correlation dimension is approximately a bilinear function of the noise level σ, and the dimension of the noise, which depends on the spectral exponent of the noise α; for larger α a wider plateau is obtained. On the other hand, the width of the scaling region for the correlation entropy depends on this spectral exponent, but the value of the plateau does not change very much. Based on our examples, we conjecture that colored noise contamination does not significantly complicate the estimation of the dimension and the entropy of chaotic systems.

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