Przemysław Figura* and Wiesław M. Macek

Dynamics of Line Preserving Field Line Motions

DOI 10.1515/zna-2015-0123
Received March 12, 2015; accepted May 11, 2015; previously published online June 5, 2015

Abstract: We develop the theory of line preserving flows and the magnetic reconnection using the Euler potentials formalism. In addition to our recently proposed model, we formulate new equations describing the time evolution of the Euler potentials in the line preserving regime. We also look at a special case of the flows represented by the ideal plasma flows. We consider the magnetic reconnection as a breakage of the line preservation regime. Because general solutions of the obtained dynamics equations do not have their closed-form expressions, we provide two different approaches to the possible solutions, in particular, a linear approximation and a solution by finding a Lagrangian and a Hamiltonian that generate the dynamics equations. We also provide some simple examples of the physical interpretation of the solutions obtained.

Keywords: Euler Potentials; Field Line Flows; Magnetic Reconnection; Magnetohydrodynamics; Plasma Physics.

PACS numbers: 52.30.Cv; 52.35.Vd.

1 Introduction

One of the very important and still not fully understood processes appearing in different fields of physics is the process of magnetic reconnection. In recent days, two methods with different mathematical tools have been used to study the processes related to the reconnection. The first, so-called, non-null (or finite-**B**) reconnection takes place in the absence of magnetic nulls or any singularities of the magnetic field (see, e.g. [1]). Whereas the second, zero-**B** reconnection is a regime of magnetic nulls, separator lines, and similar structures (see, e.g. [2]).

***Corresponding author: Przemysław Figura,** Space Research Centre, Polish Academy of Sciences, Warsaw, Poland,
E-mail: pfigura@cbk.waw.pl
Wiesław M. Macek: Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University, Warsaw, Poland; and Space Research Centre, Polish Academy of Sciences, Warsaw, Poland

Following Axford's work on the non-null reconnection [3], Schindler, Hesse, and Birn presented a concept of *the general magnetic reconnection* [4]. They used this idea to describe the non-null reconnection using the Euler potentials representation of magnetic fields [5].

We adopt here this concept and consider only the non-null magnetic reconnection. As a reconnection, we regard a process in which, during a plasma flow, the magnetic connection of plasma elements breaks down.

In the vector analysis, we can distinguish a motion during which every point starting on a given field line remains on the same line. It is called a line preserving motion, and in the case of the magnetic field **B**, it can be described by the following equation:

$$\mathbf{B}\times\left[\frac{\partial\mathbf{B}}{\partial t}-\nabla\times(\mathbf{U}_l\times\mathbf{B})\right]=0, \tag{1}$$

where $\mathbf{U}_l(\mathbf{r}, t)$ is a velocity field in a point $\mathbf{r}$ at a time t (see, e.g. [6]). The special case of the line preserving motion is a flux preserving motion. In this case, any loop moving with the velocity $\mathbf{U}_f(\mathbf{r}, t)$ preserves field's flux through the loop. This type of motion in the case of the magnetic field **B** is described by the following equation:

$$\frac{\partial\mathbf{B}}{\partial t}-\nabla\times(\mathbf{U}_f\times\mathbf{B})=0 \tag{2}$$

(see, e.g. [6]).

If we find the actual plasma flow $\mathbf{V}(\mathbf{r}, t)$ satisfying (1) or (2), i.e. if $\mathbf{V}(\mathbf{r}, t)=\mathbf{U}_l(\mathbf{r}, t)$ or $\mathbf{V}(\mathbf{r}, t)=\mathbf{U}_f(\mathbf{r}, t)$, we say that the flow $\mathbf{V}(\mathbf{r}, t)$ is line or flux preserving, respectively. We see that any flux preserving flow is always line preserving, but there are line preserving flows that do not preserve field's flux. Note that conducting fluids in the ideal plasma model are flux preserving (see, e.g. [5] or [7]). In our approach, the breakage of the flux preserving regime is not enough for the magnetic reconnection to appear. Therefore, we will say that the reconnection of the magnetic field **B** occurs if (1) is not satisfied for a given plasma flow $\mathbf{V}=\mathbf{U}_l$, i.e. if the line preserving regime is broken.

We do not consider here the zero-**B** reconnection. It can be described by the magnetic merging concept (see [8, 9]) that, as shown by Hesse and Schindler [5], however, fails in the description of the finite-**B** (i.e. $\mathbf{B}\neq 0$) reconnection. Furthermore, (1), which is the basis of our concept, in the case of zero-**B** reconnection (i.e. for $\mathbf{B}=0$), becomes just an identity. It does not contribute to our considerations besides just resembling the trivial academic fact that any two plasma elements lying on the magnetic null point are always magnetically connected by the point. Therefore, in this article, we inspect only the finite-**B** reconnection processes.

The theory of magnetic reconnection considered herein is based on the magnetohydrodynamics (MHD) theory. The proper application of MHD imposes several restrictions on the considered plasma. In particular, those limitations are the demand of the particle distribution to be close to the Maxwellian distribution; the requirement of length and time scales to be much greater than gyroradius and gyration time of plasma particles, respectively. On the other hand, collisionless systems capturing kinetic effects of the plasma are accurately described by the theory based on Vlasov's equation. Nonetheless, the convenient and simple mathematical formulation of MHD, as opposed to the kinetic approach, is the cause of the usage of MHD description in multitude instances of collisionless plasmas, especially in space and astrophysical plasmas. It can be justified, because even though in collisionless systems particles should move freely, they are still bound by their gyro-motion and unable to travel large distances perpendicularly to the magnetic field. Moreover, they are also prevented from travelling large distances along the magnetic field, because of different wave–particle interactions that usually arise in the actual plasmas. Apparently, MHD can describe, at least in a qualitative fashion, the bulk plasma flows on large scales, plasmas density, etc. Indeed, multiple studies show that observed phenomena can be accurately described using MHD, even though MHD limitations are not strictly met (e.g. for flux rope reconnection in solar corona, see [10]). A broader discussion on the applicability of MHD description to the actual plasmas can be found in a book by Priest and Forbes [11].

We use the Euler potentials representation to describe a magnetic field. For a more in-depth study on the Euler potentials, we refer to the paper by Stern [12]. In brief, we note that at least locally any divergence-free vector can be represented as a cross product of two gradients of scalar functions. For the magnetic field **B**, we have

$$\mathbf{B}=\nabla\alpha\times\nabla\beta, \tag{3}$$

where α and β are functions of coordinates x, y, and z. The magnetic vector potential **A** of the field **B** for $\mathbf{B}=\nabla\times\mathbf{A}$ is then

$$\mathbf{A}=\alpha\nabla\beta. \tag{4}$$

Thus, geometrically field lines of **B** are represented by the intersection lines of two surface families of $\alpha=\text{const.}$ and $\beta=\text{const.}$

Moreover, a choice of the potentials pair α and β is not unique, and so another potentials pair, e.g. $g(\alpha,\beta)$ and $h(\alpha,\beta)$, may represent the same magnetic field line if only

$$\frac{\partial(g,h)}{\partial(\alpha,\beta)}=1. \tag{5}$$

It is always possible to derive at least locally a set of the potentials to represent the field **B** provided that $\mathbf{B}\neq 0$ (see [13]). Moreover, we can choose at least locally a curvilinear coordinate system (α, β, s) for the given α and β families, where s is a function of x, y, and z that represents an arc length on the magnetic field line.

Despite all the limitations imposed by the application of the Euler potentials, their use is still viable. Significant results of this method can be recently found across different areas of physics from classical fluid dynamics (see, e.g. [14]), plasma physics (see, e.g. [15]), and the astrophysical simulations (see, e.g. [16]). Moreover, certain magnetic topologies may be described by the Euler potentials very conveniently. For instance, in spherical coordinates (r, θ, ϕ), a magnetic field formed by a magnetic dipole may be described by the Euler potentials pair $\alpha=\alpha_0(\sin^2\theta/r)$ and $\beta=\beta_0\phi$, where α_0 and β_0 are constants. It just illustrates the fact that the Euler potentials formalism may simplify considerations of certain magnetic field systems or shed a new light on their properties.

Naturally, the Euler potentials are still used in the magnetic reconnection domain, and the theory is continually developed. The most remarkable recent works on the subject are a paper by Hesse et al. [17] that deals with the rate of magnetic reconnection, and a paper by Wendel et al. [18] in which the authors investigate the reconnection rate and provide a method of finding places of the reconnecting field.

Nevertheless, the magnetic reconnection theory itself is continually under study. In particular, one of the interesting recent approaches to the subject is to inspect instances where both types of the reconnection exist simultaneously – non-null and zero-**B** reconnection. The configurations that are analysed combine the quasi-separatrix layer and the spine-fan reconnection (see, e.g. [19, 20]). For the review of MHD theories describing the magnetic reconnection that are currently under the most intense study, we refer to a paper by Low [21]. A review of recent advances in the magnetic reconnection theory inclined towards the kinetic approach can be found in a paper by Treumann and Baumjohann [22].

2 Dynamics Equations

Properties of the magnetic field **B**, satisfying (3), may be studied by inspecting properties of the Euler potentials α and β. Hence, dynamics of the Euler potentials determine dynamics of the magnetic field. Therefore, to develop the theory and acquire more insight into the magnetic reconnection, we examine here dynamics of the Euler potentials in the line preserving regime.

In the paper by Hesse and Schindler [5], the authors have obtained equations describing the time evolution of the Euler potentials related with the magnetic reconnection (cf. (23) from [5]). Their derivation is based on the study of Faraday's law and Ohm's law, and application of the Euler potentials formalism. We would like to derive here analogous equations describing the dynamics of the Euler potentials, but in a more general way without such a close reference to the physical setting. We base our study purely on the vector analysis' equation of the line preserving flows (cf. (1)). Thus, without the bonds of specific physical conditions, we obtain the dynamics equations that can be broadly used in any circumstances, provided that the basic vector analysis' assumptions hold; naturally, including the case of the non-null magnetic reconnection theory.

In the paper by Figura and Macek [23], we have derived equations for the line preserving field line motions of the magnetic field using the Euler potentials representation. We have obtained the following equation for the velocity field $\mathbf{U}_l$

$$\mathbf{U}\times\mathbf{B}=\frac{\partial\alpha}{\partial t}\nabla\beta-\frac{\partial\beta}{\partial t}\nabla\alpha-\zeta\alpha\nabla\beta+\nabla\Lambda, \tag{6}$$

where ζ and Λ are arbitrary functions of α and β, i.e. $\zeta=\zeta(\alpha,\beta)$ and $\Lambda=\Lambda(\alpha,\beta)$, and where index l in the velocity symbol will be omitted hereafter. Equation (6) is in fact the line preserving motion condition (cf. (1)) expressed in terms of the Euler potentials. Furthermore, the equation implies that any given line preserving flow **U** in a given magnetic field **B** can be described by appropriately chosen functions ζ and Λ. Admittedly, (6) does not allow us to determine **U** having known **B** or to determine **B** having known **U**. Actually, for the given magnetic field **B**, there might be many flows **U** that hold the line preserving condition, or conversely, for the given flow **U**, there might be different fields **B** that hold the line preserving condition.

We further investigate (6) to obtain equations of motion for the Euler potentials α and β in the line preserving regime. First, we consider the left hand side of (6), i.e. $(\mathbf{U}\times\mathbf{B})$. In the adopted Euler potentials representation, the parallel to **B** component of **U** remains undefined; therefore, for simplicity, we set it equal to 0 (see, e.g. [5] or [7]). Taking a right-hand-side dot product of the left hand side of (6) with $(\mathbf{B}\times\nabla\alpha)$, we get

$$\begin{aligned}(\mathbf{U}\times\mathbf{B})\cdot(\mathbf{B}\times\nabla\alpha)&=(\mathbf{U}\cdot\mathbf{B})(\mathbf{B}\cdot\nabla\alpha)-(\mathbf{B}\cdot\mathbf{B})(\mathbf{U}\cdot\nabla\alpha)\\&=-|\mathbf{B}|^2(\mathbf{U}\cdot\nabla\alpha),\end{aligned} \tag{7}$$

where $(\mathbf{U}\cdot\mathbf{B})=0$ for a flow perpendicular to **B**. Then, we consider the right hand side of (6), and we also take a right-hand-side dot product of it with $(\mathbf{B}\times\nabla\alpha)$. Now we get

$$\begin{aligned}&\left(\frac{\partial\alpha}{\partial t}\nabla\beta-\frac{\partial\beta}{\partial t}\nabla\alpha-\zeta\alpha\nabla\beta+\nabla\Lambda\right)\cdot(\mathbf{B}\times\nabla\alpha)\\&=\frac{\partial\alpha}{\partial t}\nabla\beta\cdot(\mathbf{B}\times\nabla\alpha)-\frac{\partial\beta}{\partial t}\nabla\alpha\cdot(\mathbf{B}\times\nabla\alpha)\\&-\zeta\alpha\nabla\beta\cdot(\mathbf{B}\times\nabla\alpha)+\nabla\Lambda\cdot(\mathbf{B}\times\nabla\alpha).\end{aligned} \tag{8}$$

From the vector calculus, we have

$$\begin{aligned}&\frac{\partial\alpha}{\partial t}\nabla\beta\cdot(\mathbf{B}\times\nabla\alpha)=\frac{\partial\alpha}{\partial t}\mathbf{B}\cdot(\nabla\alpha\times\nabla\beta)=\frac{\partial\alpha}{\partial t}|\mathbf{B}|^2,\\&\frac{\partial\beta}{\partial t}\nabla\alpha\cdot(\mathbf{B}\times\nabla\alpha)=\frac{\partial\beta}{\partial t}\mathbf{B}\cdot(\nabla\alpha\times\nabla\alpha)=0,\\&\zeta\alpha\nabla\beta\cdot(\mathbf{B}\times\nabla\alpha)=\zeta\alpha\mathbf{B}\cdot(\nabla\alpha\times\nabla\beta)=\zeta\alpha|\mathbf{B}|^2,\\&\nabla\Lambda\cdot(\mathbf{B}\times\nabla\alpha)=\mathbf{B}\cdot(\nabla\alpha\times\nabla\Lambda)=\mathbf{B}\cdot\left(\nabla\alpha\times\left(\frac{\partial\Lambda}{\partial\alpha}\nabla\alpha+\frac{\partial\Lambda}{\partial\beta}\nabla\beta\right)\right)\\&\qquad=\frac{\partial\Lambda}{\partial\beta}\mathbf{B}\cdot(\nabla\alpha\times\nabla\beta)=\frac{\partial\Lambda}{\partial\beta}|\mathbf{B}|^2,\end{aligned}$$

so combining (7) and (8), we obtain

$$-|\mathbf{B}|^2(\mathbf{U}\cdot\nabla\alpha)=\frac{\partial\alpha}{\partial t}|\mathbf{B}|^2-\zeta\alpha|\mathbf{B}|^2+\frac{\partial\Lambda}{\partial\beta}|\mathbf{B}|^2, \tag{9}$$

which is

$$\mathbf{U}\cdot\nabla\alpha+\frac{\partial\alpha}{\partial t}=-\frac{\partial\Lambda}{\partial\beta}+\zeta\alpha. \tag{10}$$

Thus, we have derived the equation for a total time derivative of $\alpha(x,y,z)$:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=-\frac{\partial\Lambda}{\partial\beta}+\zeta\alpha. \tag{11}$$

Following the same procedure, but considering the dot product of (6) with $(\mathbf{B}\times\nabla\beta)$, we obtain the equation for a total time derivative of $\beta(x,y,z)$:

$$\frac{\mathrm{d}\beta}{\mathrm{d}t}=\frac{\partial\Lambda}{\partial\alpha}. \tag{12}$$

Therefore, we obtain the system of equations describing the dynamics of the Euler potentials, and thus the dynamics of the magnetic field **B**, in the line preserving regime

$$\begin{aligned} \frac{\mathrm{d}\alpha}{\mathrm{d}t} &= -\frac{\partial\Lambda}{\partial\alpha} + \zeta a \\ \frac{\mathrm{d}\beta}{\mathrm{d}t} &= \frac{\partial\Lambda}{\partial\alpha} \end{aligned} . \tag{13}$$

The preceding system of equations has a nearly Hamiltonian structure. If it were strictly Hamiltonian, we would have the flux preserving field line dynamics (see [5]). We now consider what constraints are imposed on the arbitrary functions Λ and ζ in order to have the flux preserving flows, i.e. the ideal plasma. In the flux preserving regime, the dynamics of the Euler potentials have the form

$$\begin{aligned} \frac{\mathrm{d}\alpha}{\mathrm{d}t} &= -\frac{\partial G}{\partial\beta} \\ \frac{\mathrm{d}\beta}{\mathrm{d}t} &= \frac{\partial G}{\partial\alpha} \end{aligned} . \tag{14}$$

where G is an arbitrary function of α and β, i.e. $G = G(\alpha, \beta)$. Therefore, Λ and ζ should satisfy the system of equations

$$\begin{aligned} -\frac{\partial\Lambda}{\partial\beta} + \zeta\alpha &= -\frac{\partial G}{\partial\beta} \\ \frac{\partial\Lambda}{\partial\alpha} &= \frac{\partial G}{\partial\alpha} \end{aligned} . \tag{15}$$

Solving the system of (15), we find general constraints on Λ and ζ, namely

$$\begin{aligned} \Lambda &= G + \int g(\beta)\mathrm{d}\beta + k \\ \zeta &= \frac{g(\beta)}{\alpha} \end{aligned} , \tag{16}$$

where $g(\beta)$ is an arbitrary function of β, and k is a constant.

Substituting Λ and ζ from (16) into (13), we obtain the flux preserving regime dynamics of (14). Also, the condition for line preserving flows (1) reduces to the flux preserving flows condition (2) (see [23]). Moreover, (6) describing the line preserving flows becomes

$$\mathbf{U} \times \mathbf{B} = \frac{\partial\alpha}{\partial t}\nabla\beta - \frac{\partial\beta}{\partial t}\nabla\alpha + \nabla G, \tag{17}$$

which is an equation for the case of the flux preserving flows obtained by Vasyliunas (see [7]). Therefore, for ideal plasma flows Λ and ζ must satisfy (16).

3 Solutions of Dynamics Equations

In the line preserving regime, closed-form solutions of the Euler potentials dynamics system of (13) do not exist. We would like to present here two different approaches to finding special case solutions, which may be interesting from a physical point of view. Although all of them will require some additional assumptions, we will try to make them physically reasonable.

3.1 Linear Approximation Solution

In the first approach, we expand functions Λ and ζ in the neighbourhood of a reference point (α_0, β_0, t_0) into the Taylor series. Then, we truncate the series at the first two terms. Substituting functions Λ and ζ in this form into (13) and leaving only up to linear terms, we obtain

$$\begin{aligned} \frac{\mathrm{d}\alpha}{\mathrm{d}t} &= -K_\beta + \alpha C \\ \frac{\mathrm{d}\beta}{\mathrm{d}t} &= K_\alpha \end{aligned} , \tag{18}$$

where K_α, K_β, and C are the following constants:

$$K_\alpha = \left.\frac{\partial\Lambda}{\partial\alpha}\right|_{\alpha_0,\beta_0,t_0},$$
$$K_\beta = \left.\frac{\partial\Lambda}{\partial\beta}\right|_{\alpha_0,\beta_0,t_0},$$
$$C = \zeta(\alpha_0, \beta_0, t_0) - \alpha_0\left.\frac{\partial\zeta}{\partial\alpha}\right|_{\alpha_0,\beta_0,t_0} - \beta_0\left.\frac{\partial\zeta}{\partial\beta}\right|_{\alpha_0,\beta_0,t_0} - t_0\left.\frac{\partial\zeta}{\partial t}\right|_{\alpha_0,\beta_0,t_0}.$$

Solving system of (18) yields

$$\begin{aligned} \alpha(t) &= D_\alpha e^{Ct} + \frac{K_\beta}{C}, \\ \beta(t) &= K_\alpha t + D_\beta \end{aligned} \tag{19}$$

where D_α and D_β are constants.

We see that one of the Euler potentials, α, has an exponential behaviour in time, while the second, β, has a linear behaviour in time. It can be shown that, following the same procedure but in the case of the flux preserving flows, both potentials depend on time linearly. It means that, under the given assumptions, the line preserving flows have stronger time dependence than ideal plasma flows. We also note that beside some trivial cases like, e.g. $\Lambda = 0$, the solution is not stationary and does not converge to any constant.

The considered approximation may have its direct physical interpretation as a simple non-uniformly accelerated plane flow in a magnetic field. Considering (10) and its analogue for β, we have

$$\mathbf{U} \cdot \nabla\alpha + \frac{\partial\alpha}{\partial t} = -K_\beta + \alpha C, \tag{20}$$

$$\mathbf{U}\cdot\nabla\beta+\frac{\partial\beta}{\partial t}=K_{\alpha}. \tag{21}$$

Supposing now for simplicity a steady and uniform magnetic field $\mathbf{B}=[B_x, 0, 0]$, we see that $\partial\alpha/\partial t=0$ and $\partial\beta/\partial t=0$, so we obtain

$$\mathbf{U}\cdot\nabla\alpha=-K_{\beta}+\alpha C, \tag{22}$$

$$\mathbf{U}\cdot\nabla\beta=K_{\alpha}. \tag{23}$$

We may set as well a transformation between (α, β, s) and (x, y, z) coordinate systems, which we choose to have the same origin, as

$$\begin{cases}\alpha=ay\\ \beta=bz,\\ s=rx\end{cases} \tag{24}$$

where a, b, and r are constants. It leads us to the equations for components of the velocity $\mathbf{U}$

$$u_y=yC-\frac{K_{\beta}}{a}, \tag{25}$$

$$u_z=\frac{K_{\alpha}}{b}, \tag{26}$$

where component u_x, which is a velocity component along magnetic field lines, remains undefined (cf. Section 2). Setting constants appropriately and expressing the velocity as a function of time, we finally obtain

$$\mathbf{U}=[0, DC\exp(Ct), 0], \tag{27}$$

where D is a constant. Therefore, supposing steady and uniform magnetic field $\mathbf{B}$, we have that the considered linear approximation is a strict solution for the exponentially accelerated flows of type (27). Moreover, the approximation of the system of (13) may be used to describe other flows as long as they are similar to (27) possibly in specific regions. In particular, convenient conditions might appear on the early stages of the coronal magnetic loops expansion (see, e.g. [24] or [25]) or after applying the magnetic field to the case of an accelerated jet in a crossflow (see, e.g. [26] or [27]).

3.2 Series Expansion Separable Solution

In this approach, our goal is to find a Lagrangian and then a Hamiltonian that generate the dynamics system of (13). We suppose here that $\zeta=$const. and that Λ is separable and can be expressed as a product of three functions depending on a single variable only, namely

$$\Lambda(\alpha, \beta, t)=k(\alpha)l(\beta)r(t), \tag{28}$$

where functions $k(\alpha)$ and $l(\beta)$ have a form of certain series as given in the following.

Because of the non-conservative friction-like nature of the system of (13), they cannot be obtained from a Lagrangian with only integer derivatives (see, e.g. [28]). However, with fractional mechanics using fractional derivatives, we can formulate a Lagrangian that will allow us to derive the system of (13).

The fractional calculus is not commonly known amongst mathematicians and physicists. A proper review on the topic can be found in, e.g. [29] or an in-depth study in, e.g. [30], whereas fractional mechanics are described in, e.g. [28]. We will present here only a short summary on the subject.

We use Oldham and Spanier [30] notation here. Following Riewe [28], we define a fractional integral of order ν by

$$\frac{\mathrm{d}^{-\nu}f(t)}{\mathrm{d}(t-t_0)^{-\nu}}=\frac{1}{\Gamma(\nu)}\int_{t_0}^{t}(t-t')^{\nu-1}f(t')\mathrm{d}t' \tag{29}$$

for $\mathrm{Re}(\nu)>0$, where Γ is the gamma special function. Now provided that n is the smallest integer greater than Re(u), for $\nu=n-u$, we define a fractional derivative of order u by

$$\frac{\mathrm{d}^{u}f(t)}{\mathrm{d}(t-t_0)^{u}}=\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\frac{\mathrm{d}^{-\nu}f(t)}{\mathrm{d}(t-t_0)^{-\nu}}. \tag{30}$$

We see that if u is an integer, we obtain an ordinary derivative. Moreover, if f is suitably differintegrable then the composition rule

$$\frac{\mathrm{d}^{\mu}}{\mathrm{d}(t-t_0)^{\mu}}\frac{\mathrm{d}^{\nu}}{\mathrm{d}(t-t_0)^{\nu}}f(t)=\frac{\mathrm{d}^{\mu+\nu}}{\mathrm{d}(t-t_0)^{\mu+\nu}}f(t) \tag{31}$$

is satisfied provided that $\nu\leq 0$ or $\mu\nu\geq 0$ (see [30]).

We adopt the fractional mechanics following Riewe (see [28, 29]). The Lagrangian is a function of time t, coordinates x_j (where $j=1, \dots, J$), and time derivatives of x_j of any positive real order. If the Lagrangian is a function of N different derivatives of x_j, then by $s(n)$ (where $n=1, \dots, N$) we will denote the order of the n-th derivative of x_j. We define $s(0)=0$ to describe not derived coordinate, i.e. x_j. For example, if a given Lagrangian depends on time derivatives of x_j of order 1/2 and 5, then $N=2$ and s for coordinate x_j will have following values $s(0)=0$, $s(1)=1/2$, and $s(2)=5$.

In the fractional calculus, the application of the variational principle over time $t\in[t_a, t_b]$ to the integral

$J=\int_{t_a}^{t_b} L\,\mathrm{d}t$, where L is a Lagrangian, yields the dependence of the Lagrangian on two types of derivatives

$$q_{j,s(n)}=q_{j,s(n),t_b}=\frac{\mathrm{d}^{s(n)}x_j}{\mathrm{d}(t-t_b)^{s(n)}} \tag{32}$$

and

$$q_{j,s'(n),t_a}=\frac{\mathrm{d}^{s'(n)}x_j}{\mathrm{d}(t-t_a)^{s'(n)}} \tag{33}$$

(see [29]). Euler–Lagrange equations for the Lagrangian L have the following form:

$$\sum_{n=0}^{N}(-1)^{s(n)}\frac{\mathrm{d}^{s(n)}}{\mathrm{d}(t-t_a)^{s(n)}}\frac{\partial L}{\partial q_{j,s(n),t_b}} + \sum_{n=0}^{N'}(-1)^{s'(n)}\frac{\mathrm{d}^{s'(n)}}{\mathrm{d}(t-t_b)^{s'(n)}}\frac{\partial L}{\partial q_{j,s'(n),t_a}}=0. \tag{34}$$

Defining momenta for $n=0, \ldots, N-1$

$$p_{j,s(n)}=p_{j,s(n),t_b}=\sum_{k=0}^{N-n-1}(-1)^{s(k+n+1)-s(n+1)} \times\frac{\mathrm{d}^{s(k+n+1)-s(n+1)}}{\mathrm{d}(t-t_a)^{s(k+n+1)-s(n+1)}}\left(\frac{\partial L}{\partial q_{j,s(k+n+1),t_b}}\right) \tag{35}$$

and for $n=0, \ldots, N'-1$

$$p_{j,s'(n),t_a}=\sum_{k=0}^{N'-n-1}(-1)^{-[s'(k+n+1)-s'(n+1)]} \times\frac{\mathrm{d}^{s'(k+n+1)-s'(n+1)}}{\mathrm{d}(t-t_b)^{s'(k+n+1)-s'(n+1)}}\left(\frac{\partial L}{\partial q_{j,s'(k+n+1),t_a}}\right), \tag{36}$$

we have that Hamiltonian H has the form

$$H=\sum_{n=1}^{N}\sum_{j=1}^{J}q_{j,s(n),t_b}p_{j,s(n-1),t_b}+\sum_{n=1}^{N'}\sum_{j=1}^{J}q_{j,s'(n),t_a}p_{j,s'(n-1),t_a}-L. \tag{37}$$

For the simplicity of our study, we will consider Lagrangians depending on generalised coordinates of type (32) only. It is justified if we are considering a limiting case $t_a \to t_b$ for $t_a<t_b$, and thus, we can approximate all fractional derivatives by $\mathrm{d}^u/\mathrm{d}(t-t_b)^u$ type derivatives (see [29]). Therefore, Euler–Lagrange (34) reduce to

$$\sum_{n=0}^{N}(-1)^{s(n)}\frac{\mathrm{d}^{s(n)}}{\mathrm{d}(t-t_a)^{s(n)}}\frac{\partial L}{\partial q_{j,s(n)}}=0, \tag{38}$$

and Hamiltonian (37) reduces to

$$H=\sum_{n=1}^{N}\sum_{j=1}^{J}q_{j,s(n)}p_{j,s(n-1)}-L. \tag{39}$$

Having done this short summary, we consider now the following Lagrangian:

$$L=i\alpha^2_{\left(\frac{1}{2}\right)}+i\beta^2_{\left(\frac{1}{2}\right)}+\zeta\alpha^2-2\alpha\frac{\partial\Lambda}{\partial\beta}+2\beta\frac{\partial\Lambda}{\partial\alpha}, \tag{40}$$

where $\alpha_{(1/2)}$ and $\beta_{(1/2)}$ are fractional time derivatives of order 1/2 of α and β, respectively, and i is an imaginary unit. In generalised coordinates, we can express Lagrangian (40) as

$$L=iq^2_{\alpha,\frac{1}{2}}+iq^2_{\beta,\frac{1}{2}}+\zeta q^2_{\alpha,0}-2q_{\alpha,0}\frac{\partial\Lambda}{\partial q_{\beta,0}}+2q_{\beta,0}\frac{\partial\Lambda}{\partial q_{\alpha,0}}, \tag{41}$$

where $q_{\alpha,s(n)}$ and $q_{\beta,\,s(n)}$ are time derivatives of α and β, respectively, of order $s(n)$ as given in (32). In this particular case for the generalised coordinate q_α, the function s has values $s(0)=0$ and $s(1)=1/2$, whereas for the coordinate q_β the function s has similar values $s(0)=0$ and $s(1)=1/2$.

Making use of (31), one can obtain from Lagrangians with fractional derivatives the friction-like terms in equations of motion. In the considered case, we obtain the following Euler–Lagrange equations:

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}t}-\zeta\alpha+\frac{\partial\Lambda}{\partial\beta}-\beta\frac{\partial^2\Lambda}{\partial\alpha^2}+\alpha\frac{\partial^2\Lambda}{\partial\alpha\partial\beta}=0\\ \frac{\mathrm{d}\beta}{\mathrm{d}t}-\frac{\partial\Lambda}{\partial\alpha}+\alpha\frac{\partial^2\Lambda}{\partial\beta^2}-\beta\frac{\partial^2\Lambda}{\partial\alpha\partial\beta}=0\end{aligned}. \tag{42}$$

To recover our dynamics system of (13), we see that the following equations must be satisfied:

$$\begin{aligned}\beta\frac{\partial^2\Lambda}{\partial\alpha^2}-\alpha\frac{\partial^2\Lambda}{\partial\alpha\partial\beta}=0\\ \alpha\frac{\partial^2\Lambda}{\partial\beta^2}-\beta\frac{\partial^2\Lambda}{\partial\alpha\partial\beta}=0\end{aligned}. \tag{43}$$

From those constraints, we derive the form of series of functions $k(\alpha)$ and $l(\beta)$ as mentioned in the beginning of Section 3.2. We see that (43) are satisfied only if functions $k(\alpha)$ and $l(\beta)$ can be expressed as

$$k(\alpha)=\sum_{m=0}^{\infty}\frac{ac^m\alpha^{4m}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{3}{4}\right)}, \tag{44}$$

$$l(\beta)=\sum_{m=0}^{\infty}\frac{bc^m\beta^{4m}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{3}{4}\right)}, \tag{45}$$

where a, b, and c are constants, and Γ is the gamma function. Applying the d'Alembert's convergence criterion, we see that series (44) and (45) are convergent.

Therefore, if Λ is separable as in (28), where $k(\alpha)$ and $l(\beta)$ are expressed as in (44) and (45), respectively, then Lagrangian (40) generates the line preserving dynamics as expressed in the system of (13).

Having Lagrangian (41), we see from (39) that under the given conditions, the Hamiltonian of the line preserving dynamics is as follows:

$$H = iq^2_{\alpha,\frac{1}{2}} + iq^2_{\beta,\frac{1}{2}} - \zeta q^2_{\alpha,0} + 2q_{\alpha,0}\frac{\partial\Lambda}{\partial q_{\beta,0}} - 2q_{\beta,0}\frac{\partial\Lambda}{\partial q_{\alpha,0}}. \quad (46)$$

Hamilton's canonical equations in the fractional mechanics are

$$\frac{\partial H}{\partial q_{j,s(n)}} = (-1)^{s(n+1)-s(n)}\frac{\mathrm{d}^{s(n+1)-s(n)}}{\mathrm{d}(t-t_a)^{s(n+1)-s(n)}}p_{j,s(n)}, \quad (47)$$

$$\frac{\partial H}{\partial p_{j,s(n)}} = q_{j,s(n+1)}, \quad (48)$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (49)$$

In the case under study, (47) recover the system of (13), whereas the remaining canonical equations are just identities.

The considered Lagrangian in (40) is obviously not the only one that generates (13) for separable Λ and constant ζ. In particular, if one specifies different forms of functions $k(\alpha)$ and $l(\beta)$, then different Lagrangians will generate the system of (13).

The physical interpretation of the presented example is as follows. For the ease of the presentation, similarly to the example from Section 3.1, we will consider a flow in a given uniform time-independent magnetic field $\mathbf{B}=[B_x, 0, 0]$. Likewise, it will simplify the transformation between (x, y, z) and (α, β, s) coordinate systems that we set to have the same origin. Then, the transformation between coordinate systems is described by

$$\begin{cases} \alpha = ay \\ \beta = bz, \\ s = rx \end{cases} \quad (50)$$

where a, b, and r are constants. We will search for the velocity field $\mathbf{U}=[u_x, u_y, u_z]$ expressed in (x, y, z) coordinate system. As stated in Section 2, the velocity along the field $\mathbf{B}$ lines is undefined; thus, in the considered case, we can set $u_x=0$. The remaining components of $\mathbf{U}$ we will derive from (6).

The flow $\mathbf{U}$ is described by the Lagrangian (40) if ζ and Λ of the flow have the following forms:

$$\begin{aligned} &\zeta = \text{const.} \\ &\Lambda = k(\alpha)l(\beta)r(t)\text{'} \end{aligned} \quad (51)$$

where $k(\alpha)$ and $l(\beta)$ are

$$\begin{aligned} k(\alpha) &= \sum_{m=0}^{\infty}\frac{D_k c^m \alpha^{4m}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\,\Gamma\left(m+\frac{3}{4}\right)} \\ l(\beta) &= \sum_{m=0}^{\infty}\frac{D_l c^m \beta^{4m}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\,\Gamma\left(m+\frac{3}{4}\right)}, \end{aligned} \quad (52)$$

with D_k, D_l, and c are constants. After the transformation to (x, y, z) coordinate system, (6) of the considered flow is

$$B_x\begin{bmatrix}0\\u_z\\-u_y\end{bmatrix} = -\zeta ay\begin{bmatrix}0\\0\\b\end{bmatrix} + r(t)\begin{bmatrix}0\\ l(bz)\frac{\partial}{\partial y}k(ay)\\ k(ay)\frac{\partial}{\partial z}l(bz)\end{bmatrix}. \quad (53)$$

Therefore, the flow U has the following components:

$$u_x = 0, \quad (54)$$

$$u_y = \zeta\frac{ab}{B_x}y - \frac{r(t)}{B_x}\sum_{m=0}^{\infty}\frac{D_k c^m a^{4m}y^{4m}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{3}{4}\right)} \times\sum_{m=0}^{\infty}\frac{D_l c^m b^{4m}4mz^{4m-1}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{3}{4}\right)}, \quad (55)$$

$$u_z = \frac{r(t)}{B_x}\sum_{m=0}^{\infty}\frac{D_l c^m b^{4m}z^{4m}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{3}{4}\right)} \times\sum_{m=0}^{\infty}\frac{D_k c^m a^{4m}4my^{4m-1}\Gamma\left(\frac{3}{4}\right)}{4^{2m}m!\Gamma\left(m+\frac{3}{4}\right)}. \quad (56)$$

Depending on the values of constants, different flows can be described by the Lagrangian (41). In Figure 1, there are presented two examples of the possible velocity vector fields of such flows in a stationary case, i.e. for $r(t)=\text{const}$. Both of them were generated using the same values of parameters except for the value of constant c, which was positive in the case of the flow from panel 1(a) and negative in the case of the flow from panel 1(b). As seen in Figure 1, significantly different flows can be generated – in the considered example a vortex-like flow (panel 1(a)) or a diffusion-like outflow (panel 1(b)). Apparently,

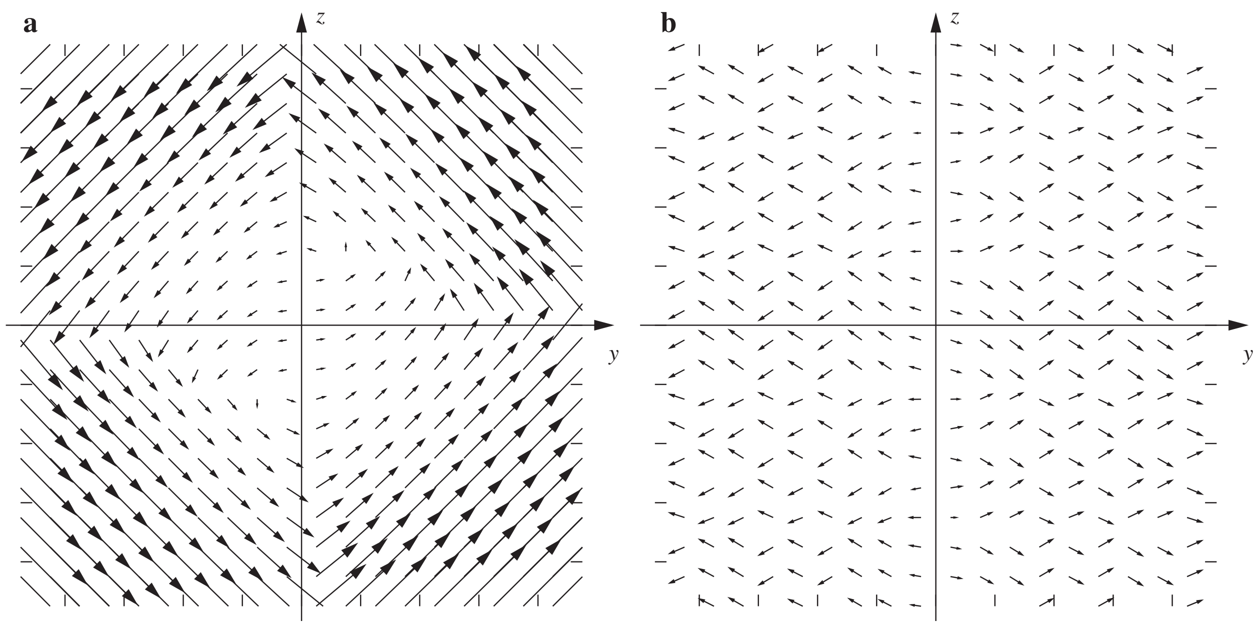

Figure 1: Examples of the velocity vector fields of flows described by the Lagrangian (41). The field **B** is perpendicular to the plane of flows. The parameters used: $\zeta=1.5$, $r=1$, $B_x=1$, $a=0.8$, $b=0.8$, $c=1.2$ – panel (a) or $c=-1.2$ – panel (b), $D_k=0.5$, and $D_l=0.5$. For the clarity of the picture, the normalised natural logarithms of the vectors were plotted individually on each panel.

this simple example shows the variety of the line preserving flows that can be generated by the same Lagrangian.

4 Conclusions

We have formulated here the dynamics of a magnetic field in the line preserving flows regime using the Euler potentials representation. In particular, not tied to any specific physical setting, but based on the vector analysis, we have derived the system of equations describing the time dependence of the Euler potentials. As a special case, we have also described the dynamics of flux preserving motions that characterise the ideal plasma flows. Our novel formulation of the magnetic field dynamics as a nearly Hamiltonian system may be useful in studies on the non-null magnetic reconnection as well as other topics concerning the line preserving flows. For instance, it may streamline considerations of the magnetic fields that have a simple Euler potentials representation. Additionally, our formulation may be used to find constraints for the existence of the possible solutions. Likewise, it may be used in the reconnection instability analysis of the various flows in magnetic fields.

In our approach, we adopted a breakage of the line preservation regime as a non-null magnetic reconnection indicator. A similar concept and its considerations using the Euler potentials have been recently examined by, e.g. Nickeler and Fahr [31]. Our results might be useful in similar studies, but also in the modelling attempts of the subject. Moreover, even though the plasma setting is used, our considerations remain valid for any flows in other vector fields satisfying the same basic assumptions, i.e. zero divergence of the field and local applicability of the Euler potentials representation. Therefore, applications of the results are much wider than to the plasma physics only.

Because of the nonexistence of closed-form solutions in the line preserving regime, we have proposed two different approaches to finding special case solutions. The first of them was a simple solution by linearisation. It appeared that the solution may strictly describe exponentially accelerated flows in a uniform, steady magnetic field. The second approach was to find a Lagrangian and a Hamiltonian for the line preserving flows dynamics. It proved to be possible for different parameters sets, and we have provided examples of two such diverse flows that can be generated by the same Lagrangian.

We think our studies may be interesting for a greater number of researchers, not only for the plasma physics specialists. In our opinion, further study of the presented model may be fruitful, especially in the magnetic reconnection domain.

Acknowledgments: This work has been supported by the European Community's Seventh Framework Programme

[FP7/20072013] under grant agreement No. 313038/ STORM. The authors thank Dr. Deirdre E. Wendel for suggestions and inspiring comments that significantly improved the manuscript.

References

[1] J. C. Dorelli and A. Bhattacharjee, Phys. Plasmas **15**, 056504 (2008).
[2] J. M. Greene, J. Geophys. Res. **93**, 8583 (1988).
[3] W. I. Axford, in: Magnetic Reconnection in Space and Laboratory Plasmas, Geophys, Monograph No. 30 (Ed. E. W. Hones, Jr.), American Geophysical Union, Washington, DC 1984, pp. 1–8.
[4] K. Schindler, M. Hesse, and J. Birn, J. Geophys. Res. **93**, 5547 (1988).
[5] M. Hesse and K. Schindler, J. Geophys. Res. **93**, 5559 (1988).
[6] D. P. Stern, J. Geophys. Res. **78**, 1702 (1973).
[7] V. M. Vasyliunas, J. Geophys. Res. **77**, 6271 (1972).
[8] V. M. Vasyliunas, Rev. Geophys. **13**, 303 (1975).
[9] V. M. Vasyliunas, in: Magnetic reconnection in Space and Laboratory Plasmas, Geophys. Monograph No. 30 (Ed. E. W. Hones, Jr.), American Geophysical Union, Washington, DC 1984, pp. 25–31.
[10] T. Török, R. Chandra, E. Pariat, P. Démoulin, B. Schmieder, G. Aulanier, M. G. Linton, and C. H. Mandrini, Astrophys. J. **728**, 65 (2011).
[11] E. Priest and T. Forbes, Magnetic Reconnection, Cambridge University Press, New York 2000.
[12] D. P. Stern, Am. J. Phys. **38**, 494 (1970).
[13] H. B. Phillips, Vector Analysis, John Wiley & Sons, Inc., New York 1933.
[14] A. Yahalom and D. Lynden-Bell, J. Fluid Mech. **607**, 235 (2008).
[15] K. K. Khurana, J. Geophys. Res. **102**, 11295 (1997).
[16] H. Kotarba, H. Lesch, K. Dolag, T. Naab, P. H. Johansson, and F. A. Stasyszyn, Mon. Not. R. Astron. Soc. **397**, 733747 (2009).
[17] M. Hesse, T. G. Forbes, and J. Birn, Astrophys. J. **631**, 1227 (2005).
[18] D. E. Wendel, D. K. Olson, M. Hesse, N. Aunai, M. Kuznetsova, H. Karimabadi, W. Daughton, and M. L. Adrian, Phys. Plasmas. **20**, 122105 (2013).
[19] S. Masson, G. Aulanier, E. Pariat, and K. -L. Klein, Solar Phys. **276**, 199 (2012).
[20] D. I. Pontin, E. R. Priest, and K. Galsgaard, Astrophys. J. **774**, 154 (2013).
[21] B. C. Low, Sci. China Phys. Mech. Astron. **58**, 015201 (2015).
[22] R. A. Treumann and W. Baumjohann, Frontiers Phys. **1**, 31 (2013).
[23] P. Figura and W. M. Macek, Ann. Phys. **333**, 127 (2013).
[24] K. Shibata, T. Tajima, and R. Matsumoto, Phys. Fluids B. **2**, 9 (1990).
[25] H. Lee and T. Magara, Publ. Astron. Soc. Japan. **66**, 39 (1) (2014).
[26] A. Eroglu and R. E. Breidenthal, AIAA J. **36**, 1002 (1998).
[27] R. E. Breidenthal, Phys. Scr. **T132**, 014001 (2008).
[28] F. Riewe, Phys. Rev. E. **55**, 3581 (1997).
[29] F. Riewe, Phys. Rev. E. **53**, 1890 (1996).
[30] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, Inc., California 1974.
[31] D. H. Nickeler and H.-J. Fahr, Solar Phys. **235**, 191 (2006).