Chaos and Multifractals in the Solar System Plasma

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We argue that dynamical behavior of space plasmas can often be approximately described by low-dimensional chaotic attractors in the inertial manifold, which is a subspace of a given system phase space. In fact, using nonlinear time-series analysis based on the method of topological embedding, we have identified a chaotic strange attractor in the solar wind data. In particular, we have shown that the multifractal spectrum of the solar wind attractor is consistent with that for the self-similar generalized weighted Cantor set with one probability measure parameter of the chaotic attractor and one or possibly two scaling parameters describing nonuniform compression in the phase space of the system. The values of the parameters fitted demonstrate small dissipation of the complex solar wind plasma and show that some parts of the attractor in phase space are visited much more frequently than other parts.

To quantify the multifractality of space plasma turbulence, we have recently considered the generalized two-scale weighted Cantor set also in the context of solar wind intermittent turbulence. We investigate the resulting multifractal spectrum of generalized dimensions depending on parameters of the new cascade model, especially for asymmetric scaling. In particular, we show that intermittent pulses are stronger for the model with two different scaling parameters; a much better agreement with the solar wind data is obtained, especially for the negative index of the generalized dimensions.

Therefore we argue that there is a need to use a two-scale cascade model. We hope that this generalized multifractal model will be a useful tool for analysis of intermittent turbulence in the Solar System plasma. We thus believe that fractal analysis of chaotic phenomena in the complex space environment could lead us to a deeper understanding of their nature, and maybe even to predict their seemingly unpredictable behavior.

Keywords: Nonlinear dynamics and chaos; Multifractals; Turbulence; Solar wind plasma.

1. Importance of Chaos and Multifractality

The nature of the fluctuations in solar wind plasma parameters is still not sufficiently understood. The slow solar wind most likely originates from nonlinear processes in the solar corona. However, it appears that a certain kind of order does lie concealed within the irregular solar wind fluctuations, which can be described using methods of nonlinear time series analysis, based on fractal analysis and the theory of deterministic chaos. This involves the notions of fractal and multifractal sets, which could presumably be strange attractors in a certain state space of a given complex dynamical system. By employing the so-called false-nearest-neighbors method, we have argued that the deterministic component of solar wind plasma dynamics should be low-dimensional.[1]

In fact, the results we have obtained using the method of topological embedding indicate that the behavior of the solar wind can be approximately described by a low-dimensional chaotic attractor in the inertial manifold, which is a subspace of system phase space. A direct determination of a solar wind attractor from the data is known to be a difficult problem. This chaotic strange attractor has been identified in the solar wind data by Macek in [2] and further examined in [3]. In particular, we have calculated the correlation dimension of the reconstructed attractor in the solar wind[2] and have provided tests for this measure of *complexity* including statistical surrogate data tests.[4] Further, we have shown that the Kolmogorov entropy of the attractor is *positive* and finite, as it holds for a *chaotic* system.[3]

We have also considered the spectrum of generalized dimensions D_q as a function of a continuous index, $-\infty < q < \infty$, for the solar wind attractor, using a simple multifractal model with a measure of the self-similar weighted Cantor set with one parameter describing uniform compression and another parameter for the probability measure of the attractor of the system. The spectrum is found to be consistent with the data, at least for positive index q of the generalized dimensions D_q.[5–9] However, the full singularity spectrum is necessary to quantify the degree of multifractality. Notwithstanding of the well-known statistical problems with negative q (see [7]) we have recently succeeded in estimating the entire spectrum for solar wind attractor using a generalized weighted Cantor set with two different scales describing nonuniform compression.[10]

The question of multifractality is also of great importance because it allows us to investigate the nature of interplanetary hydromagnetic turbulence in the solar wind e.g. [11,12]. Starting from Richardson's version of turbulence, many authors try to recover the observed scaling exponents, using some simple and more advanced models of turbulence describing distribution of the energy flux between cascading eddies at various scales. In particular, the multifractal spectrum was investigated using Voyager (magnetic field) data in the outer heliosphere[13,14] and using Helios (plasma) data in the inner heliosphere.[15] The multifractal scaling has also been investigated using Ulysses observations[16,17] and with Advanced Composition Explorer (ACE) and WIND data.[18–20]

Recently, in order to further quantify the multifractality, we have considered the generalized weighted Cantor set also in the context of turbulence cascade.[21] Therefore we have argued that there is, in fact, need to use a two-scale cascade model. Here we investigate the resulting multifractal singularity spectrum depending on two scaling parameters and one probability measure parameter, demonstrating that a much better agreement has been obtained, especially for $q < 0$. We hope that this generalized new asymmetric multifractal model could shed light on the nature of turbulence and will be a useful tool for analysis of intermittent turbulence in various environments.

2. Two-scale Cantor Set

A simple interesting example of mutifractals is the Cantor-set with two scales $l_1+l_2 \le l$, as shown in Figure 1. At each step of construction we obtain 2^n closed narrow segments of various widths and probabilities. The resulting strange chaotic attractor for $n \to \infty$ is the weighted two-scale Cantor set. Even though one can find this generalized Cantor set in many classical textbooks,[22,23] it is still difficult to understand this strange attractor that exhibits multifractality in various complex real systems, also in case of intermittent turbulence.

According to a standard scenario, each of cascading eddies is breaking down into two new ones, but not necessarily equal and twice smaller. In particular, space filling turbulence could be recovered for $l_1 + l_2 = 1$ (see [24]). In the inertial region of the system of size L, $\eta \ll l \ll L = 1$ (normalized), the energy is not allowed to be dissipated directly until the Kolmogorov scale η is reached. However, in this range at each n-th step of the binomial multiplicative process, the flux of kinetic energy density ε transferred to smaller eddies (energy transfer rate) could be divided into non equal fractions p and $1 - p$ (see [21], Figure 1).

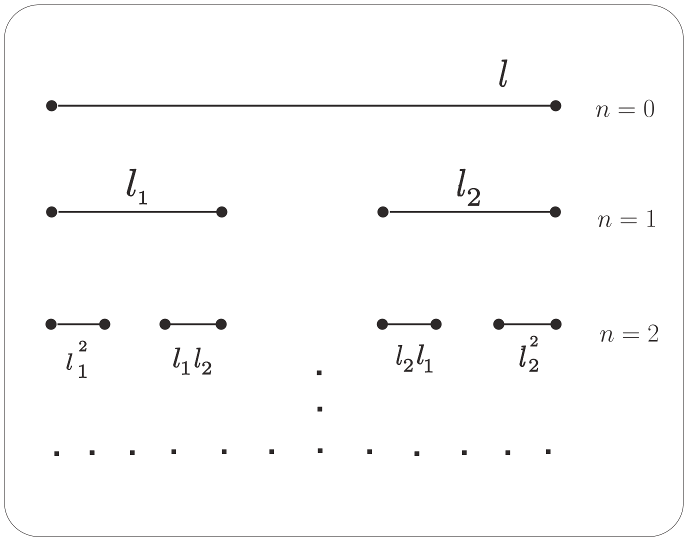

Fig. 1. Two-scale Cantor set.

3. Solar Wind Data

We have already analyzed the Helios 2 data using plasma parameters measured *in situ* in the inner heliosphere[25] for testing of the solar wind attractor. The X-velocity (mainly radial) component of the plasma flow, v_x, has been already investigated by Macek[2,5,6] and Macek and Redaelli.[3] The Alfvénic fluctuations with longer (two-days) samples have been studied by Macek[7,10] and Macek et al.[8,9] To study the turbulence cascade, Macek and Szczepaniak[21] have selected four-day time intervals of v_x samples in 1976 (solar minimum) for both slow and fast solar wind streams measured at various distances from the Sun. In this paper we analyze time series of velocities of the solar wind measured by ACE in the ecliptic plane near the libration point $L1$, e.g., approximately at a distance of $R = 1$ AU from the Sun. Here we have selected even longer (five-day) time intervals of v_x samples, each of 6750 data points, interpolated with sampling time of 64 s, for both slow and fast solar wind streams during solar minimum (2006) and maximum (2001).

4. Methods of Data Analysis

4.1. *Generalized Dimensions*

The generalized dimensions are important characteristics of *complex* dynamical systems.[26–29] Since these dimensions are related to frequencies with which typical orbits in phase space visit different regions of the attractors, they provide information about dynamics of the systems.[23] More precisely, one may distinguish a probability measure from its geometrical support, which may or may not have fractal geometry. Then, if the measure has different fractal dimensions on different parts of the support, the measure is multifractal.[30]

Let us consider the generalized weighted Cantor set where the probability of visiting one segment of size l_1 is p (say, $p \leq 1/2$), and for the remaining segment of size l_2 is $1 - p$ (see Figure 1). For any q one obtains $D_q = \tau(q)/(q-1)$ by solving numerically the following transcendental equation[23]

$$\frac{p^q}{l_1^{\tau(q)}} + \frac{(1-p)^q}{l_2^{\tau(q)}} = 1 \tag{1}$$

4.2. *Turbulence Scaling*

In the inertial range the transfer rate of the energy flux $\varepsilon(l)$ is widely estimated by

$$\varepsilon(l) \sim \frac{\left|u(x+l) - u(x)\right|^3}{l}, \tag{2}$$

where $u(x)$ and $u(x+l)$ are velocity components parallel to the longitudinal direction separated from a position x by a distance l. Therefore to each ith eddy of size l in the turbulence cascade ($i = 1, \ldots, N = 2^n$) we associate a probability measure defined by

$$p_i(l) = \frac{\varepsilon_i(l)}{\sum_{i=1}^{N} \varepsilon_i(l)} \tag{3}$$

This quantity can roughly be interpreted as a probability that the energy flux is transferred to an eddy of size $l = v_{\rm sw}t$. Here, for simplicity the third moment of structure function of velocity fluctuations in Eq. (2) is used for estimation of this measure.[15]

Similarly, we define a one parameter q family of (normalized) generalized pseudoprobability measures[31]

$$\mu_i(q, l) \equiv \frac{p_i^q(l)}{\sum_{i=1}^{N} p_i^q(l)} \tag{4}$$

Now, with an associated fractal dimension index $f_i(q,l) \equiv \log \mu_i(q,l)/\log l$ for a given q the multifractal singularity spectrum of dimensions is defined directly as the averages taken with respect to the measure $\mu(q,l)$ in Eq. (4) denoted from here on by $\langle \ldots \rangle$

$$f(q) \;\equiv\; \lim_{l\to 0} \sum_{i=1}^{N} \mu_i(q,l)\; f_i(q,l) = \lim_{l\to 0} \frac{\big\langle \log \mu_i(q,l) \big\rangle}{\log(l)} \tag{5}$$

and the corresponding average value of the singularity strength is given by[32]

$$\alpha(q) \;\equiv\; \lim_{l\to 0} \sum_{i=1}^{N} \mu_i(q,l)\; \alpha_i(l) = \lim_{l\to 0} \frac{\big\langle \log p_i(l) \big\rangle}{\log(l)} \tag{6}$$

One can easily verify that the multifractal singularity spectrum $f(\alpha)$ as a function of α satisfies the following Legendre transformation[29,32]

$$\alpha(q) \;=\; \frac{\mathrm{d}\tau(q)}{\mathrm{d}q}, \qquad f(\alpha) \;=\; q\alpha(q) - \tau(q). \tag{7}$$

5. Results and Discussion

5.1. *Dimensions for Solar Wind Attractor and Turbulence Models*

To estimate the generalized dimensions for the solar wind attractor we should calculate for a given continuous index q and embedding dimension m the so-called generalized correlation sum $C_{q,m}(r)$ as a function of hyperspheres of radius r that cover the presumed attractor (see [10], Equation 1). This can roughly be interpreted as an average probability of finding q vectors in embedding space separated by a distance smaller than r. For large dimensions m and small distances r in the scaling region it can be argued that $C_{q,m}(r) \propto r^{\tau(q)}$, where $\tau(q)$ is an approximation of the ideal solution of Eq. (1).[27] Hence, the slopes of the natural logarithm of $C_{q,m}(r)$ versus $\ln r$ (normalized) provides

$$D_{q,m}(r) = \frac{1}{q-1}\,\frac{\mathrm{d}\big[\ln C_{q,m}(r)\big]}{\mathrm{d}(\ln r)} \tag{8}$$

If a plateau exists in a scaling region, $r_{\min} < r < r_{\max}$, which does not depend on m for some $m > m_0$, this plateau can be identified with the requested generalized dimension. Finally, the average slope for $6 \le m \le 10$ is taken as D_q.[2,7–10] We have verified that for $q > 0$ the slopes do not change substantially with the number of points used, providing that the dimension of the attractor is well below $2\log_{10} N \approx 9$, e.g., for $N = 26163$ (see [33]). The results obtained using the moving average filter and singular-value decomposition linear filter for standard $q = 2$ are given in Figure 2 of [8], and are compared with $q = -2$ in Figures 2 (a) and (b) of [7], correspondingly, while those obtained for somewhat shorter samples ($N = 4514$) have been discussed in [2] and [3], using the nonlinear Schreiber filters. Next, the generalized dimensions D_q as a function of q (see [10], Equation 1) with the statistical errors of the average slopes obtained using weighted least squares fitting over the scaling range are shown in Figures 3 (a) and 4 (a) of [10] and compared with one-scale and two-scale Cantor set model (see [8], Figure 3).

5.2. *Multifractal Spectrum for Turbulence*

The results for the generalized dimensions D_q as a function of q, calculated from the data and compared with those obtained using Eq. (1) for solar wind turbulence for the slow (a) and (c) and fast (b) and (d) solar wind streams at distances of 0.3 AU and 0.97 AU, correspondingly, are presented in Figures 3 (a), (b), (c) and (d) of [21].

Here the results for the corresponding singularity spectra $f(\alpha)$ as a function of α are shown in Figures 2 (a), (b), (c) and (d). The values of $f(\alpha)$ given in Eqs. (5) and (6), for one-dimensional turbulence, $d = 1$, are calculated using the radial velocity components $u = v_x$ (in time domain), cf. Figure 3 of [21]. It is well known that for $q < 0$ we have some basic statistical problems.[7,10] Nevertheless, in spite of large statistical errors in Figure 2, especially for $q < 0$, the multifractal character of the measure can still clearly be discerned. Therefore one can confirm that both the spectrum of dimensions and singularity spectrum still exhibit the multifractal structure of the solar wind in the inner heliosphere.[21]

For $q \geq 0$ these results agree with the usual one-scale p-model fitted to the singularity spectra as obtained analytically using $l_1 = l_2 = 0.5$ in Eq. (1) and the corresponding value of the parameter $p = 0.21$ and 0.20, 0.15 and 0.12 for the slow (a) and (c), and fast (b) and (d) solar wind streams at solar minimum and maximum, correspondingly, as shown by dashed lines. On the contrary, for $q < 0$ the p-model cannot describe the observational results.[15] Here we show that the experimental values are consistent also with the singularity spectrum obtained numerically from Eqs. (5-6) for the weighted two-scale Cantor set using an asymmetric scaling, i.e., using unequal scales $l_1 \neq l_2$, as is shown in Figures 2 (a), (b), (c), and (d) by continuous lines. We also confirm that the degree of multifractality and asymmetry of the

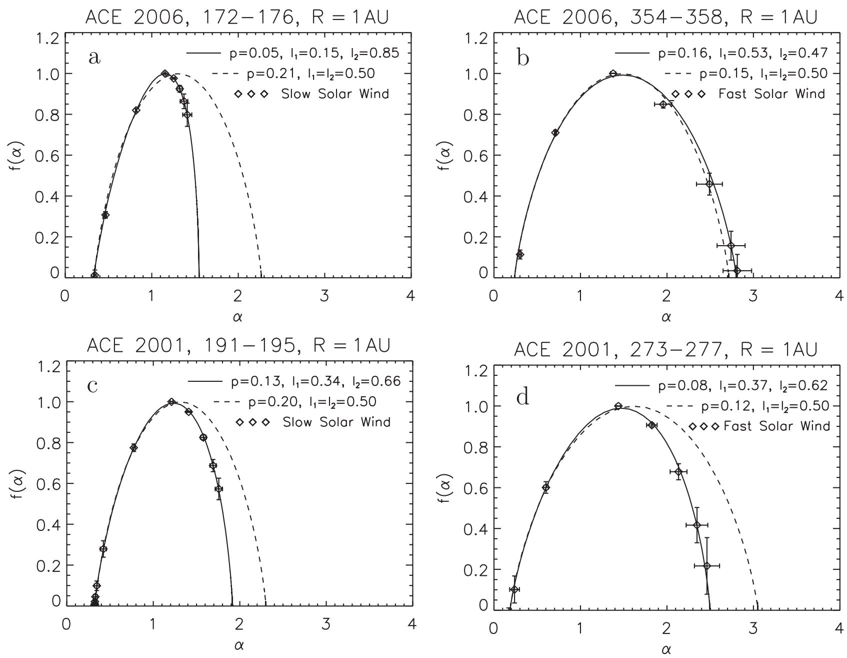

Fig. 2. The singularity spectrum $f(\alpha)$ as a function of α. The values obtained for one-dimensional turbulence are calculated for the usual one-scale (dashed lines) p-model and the generalized two-scale (continuous lines) model with parameters fitted to the multifractal measure $\mu(q,l)$ obtained using the v_x velocity components measured by ACE at 1 AU (diamonds) for the slow (a) and (c) and fast (b) and (d) solar wind during solar minimum (2006) and maximum (2001), correspondingly.

solar wind in the inner heliosphere are different for slow and fast streams. One can say that in the slow streams the scaling is more asymmetric than those for the fast wind. It also seems that the degree of asymmetry for the slow wind is rather anticorrelated with the phase of the solar activity.

We see that the multifractal spectrum of the solar wind is only roughly consistent with that for the multifractal measure of the self-similar weighted symmetric one-scale weighted Cantor set only for $q \geq 0$. On the other hand, this spectrum is in a very good agreement with two-scale asymmetric weighted Cantor set schematically shown in Figure 1 for both positive and negative q. Obviously, taking two different scales for eddies in the cascade, one obtains a more general situation than in the usual p-model for fully developed turbulence,[34] especially for an asymmetric scaling, $l_1 \neq l_2$. Hence we hope that this generalized model will be a useful tool for analysis of intermittent turbulence in space plasmas.

6. Conclusions

In this way, we have supported our conjecture that trajectories describing the system in the inertial manifold of phase space asymptotically approach the attractor of low-dimension.[2] We have shown that the multifractal spectrum of the solar wind attractor is consistent with that for the multifractal measure of the generalized two-scale weighted Cantor set. The values of the parameters fitted for $l_1 + l_2 = 1$ and $p \sim 10^{-1}$, for the slow wind, demonstrate small dissipation of the complex solar wind plasma and show that some parts of the attractor in phase space are visited at least one order of magnitudes more frequently than other parts (see Figure 5 of [2]).

We have also studied the inhomogeneous rate of the transfer of the energy flux indicating multifractal and intermittent behavior of solar wind turbulence in the inner heliosphere. In particular, we have demonstrated that a much better agreement with the real data is obtained, especially for $q < 0$. Basically, the generalized dimensions for solar wind are consistent with the generalized p-model for both positive and negative q, but rather with different scaling parameters for sizes of eddies, while the usual p-model can only reproduce the spectrum for $q \geq 0$. In general, the proposed generalized two-scale weighted Cantor set model should also be valid for non space filling turbulence. Therefore we propose this cascade model describing intermittent energy transfer for analysis of turbulence in various environments.

Thus these results show multifractal structure of the solar wind in the inner heliosphere. Hence we suggest that there exists an inertial manifold for the solar wind, in which the system has *multifractal* structure, and where noise is certainly not dominant. The multifractal structure, convected by the wind, might probably be related to the complex topology shown by the magnetic field at the source regions of the solar wind.

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